STABILITY REGION OF SINGULARLY PERTURBED SYSTEMS

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Abstract—This paper studies the characterization of the stability boundary (the topological boundary of stability region) of nonlinear autonomous singularly perturbed dynamical systems. In particular, a relationship between the stability boundary of the singularly perturbed system and the stability boundary and stability region of the slow and fast systems is investigated. Conditions, in terms of location and stability properties of equilibriums of the slow and fast system, for ensuring that a type-one equilibrium point lies on the stability boundary of the singularly perturbed system are derived. It is also shown how approximations of compact subsets of the stability boundary of the singularly perturbed system can be obtained in terms of the stability boundary and stability regions of the fast and slow systems.

Resumo – Este artigo estuda a caracterização da fronteira da região de estabilidade de sistemas dinâmicos autônomos não lineares singularmente perturbados. Em particular, uma relação entre a fronteira da região de estabilidade do sistema singularmente perturbado e a região de estabilidade e fronteira da região de estabilidade dos subsistemas lento e rápido é investigada. Condições, em termos da localização e propriedades dos equilíbrios dos sistemas rápido e lento, para garantir que um ponto de equilíbrio hiperbólico do tipo-1 pertença à fronteira da região de estabilidade do sistema singularmente perturbado são exibidas. Mostra-se também como aproximações de subconjuntos compactos da fronteira da região de estabilidade do sistema singularmente perturbado podem ser obtidas a partir da fronteira e região de estabilidade dos sistemas rápido e lento.

Keywords—Stability Region, Stability, Attraction Basin, Singularly perturbed systems.

1 Introduction

Many nonlinear physical systems have multi-time scale features in their state variables (Peponides et al., 1982; Chow, 1982 and Alberto & Chiang, 2008, Kokotovic et al., 1999). Several advantages in computational speed and accurate results can be derived from these features by decomposing the analysis of the original system into the analysis of two simpler systems: the slow and fast systems. These advantages include: (i) speeding up computational estimation of solutions, (ii) more well-conditioned numerical algorithms, (iii) more insightful information of the two-time-scale system by the analysis of reduced simpler subsystems.

For a particular class of two-time-scale systems, denominated singularly perturbed systems, Tikhonov has derived conditions for ensuring the proximity between their solutions and solutions of the slow and fast systems (O’Malley, 1991, page 46). This theory was further extended in (Vasil’eva & Butuzov, 1973). Stability analysis of a class of singularly perturbed systems was decomposed into the stability analysis of the slow and fast systems using Lyapunov based methods in (Klimushchev & Krasovskii, 1961). More recent contributions along this line of advance can be found in (Saberi & Khalil, 1984).

In this paper, we decompose the stability region and stability boundary characterization of singularly perturbed systems into stability region and stability boundary of the slow and fast subsystems. A comprehensive theory for stability regions of classes of nonlinear dynamical systems can be found, for example, in Chiang et al. (1988) and Venkatasubramanian et al. (1995). Recent advances in estimating stability regions can be found, for example, in Tan & Packard, (2008) and Halsey & Glover, (2005). However, the existing body of theory related to the characterization of stability region (regions of attraction) does not take into account multi-time scale properties of dynamical systems. Several advantages can be achieved by taking into account two-time-scale properties.

In this paper, the relationship among the stability boundary (the topological boundary of stability region) of the TTS system and the stability boundaries of the slow and fast systems will be developed. It will be shown that compact subsets of the stability boundary of the TTS system can be sufficiently approximated by subsets of the stability boundary and stability region of slow and fast systems.

2 Mathematical Preliminaries

Consider the following general form of singularly perturbed systems:
\( (\Sigma_{\varepsilon}) \) \[ \begin{align*} \dot{x} &= f(x,z) \\ \dot{z} &= g(x,z) \end{align*} \] (1)

where \( x \in \mathbb{R}^n \), \( z \in \mathbb{R}^m \), functions \( f \) and \( g \) are of class \( C^1 \) and \( \varepsilon \) is a positive small real parameter. Consider also its corresponding version in the fast time scale \( t=\varepsilon \tau \):
\[ \left( \Pi_{\varepsilon} \right) \] \[ \frac{dx}{d\tau} = \varepsilon f(x,z) \quad \frac{dz}{d\tau} = g(x,z) \] (2)

A point \((x^*,z^*)\) is an equilibrium point of \((\Sigma_{\varepsilon})\) if \( f(x^*,z^*)=0 \) and \( g(x^*,z^*)=0 \). An equilibrium point is hyperbolic if all eigenvalues of the Jacobian matrix \( J \) calculated at the equilibrium point do not lie on the imaginary axis. A hyperbolic equilibrium point is a type-\( k \) equilibrium point if there exist exactly \( k \) eigenvalues of \( J \) on the right-half of the complex plane. The stable and unstable manifolds of a hyperbolic equilibrium point \((x^*,z^*)\) of \((\Sigma_{\varepsilon})\) will be respectively denoted \( W^s_{\varepsilon}(x^*,z^*) \) and \( W^u_{\varepsilon}(x^*,z^*) \). The subindex \( \varepsilon \) will be used to indicate that these are invariant manifolds of \((\Sigma_{\varepsilon})\).

Let \( \varphi(t,x^0,z^0) \) denote the trajectory of \((\Sigma_{\varepsilon})\) starting at \((x^0,z^0)\). Suppose \((x^0,z^0)\) is an asymptotically stable equilibrium point (SEP) of \((\Sigma_{\varepsilon})\), then the set \( A_\varepsilon(x^0,z^0) = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \varphi(t,x^0,z^0) \rightarrow (x,z) \text{ as } t \rightarrow \infty \} \) denoting the collection of all initial conditions of \((\Sigma_{\varepsilon})\) whose trajectories converge to \((x^0,z^0)\) as \( t \rightarrow \infty \), exists and is unique. This set is termed the stability region (or region of attraction) of the SEP \((x^0,z^0)\). Our next interest is to study the behavior of \( A_\varepsilon \) as \( \varepsilon \rightarrow 0 \) and its relationship with the stability region \( A_{\varepsilon}(x^0,z^0) \) of the slow system:
\[ (\Sigma_0) \] \[ \begin{align*} \dot{x} &= f(x,z) \\ 0 &= g(x,z) \end{align*} \] (3)

and with the stability regions \( A_F(x,z) = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \varphi(t,x,z) \rightarrow (x,z) \text{ as } t \rightarrow \infty \} \) of a family of fast systems (also known as boundary layer systems):
\[ (\Pi_F(x)) \] \[ \frac{dx}{d\tau} = g(x,z) \] (4)

where \( x \) is considered a frozen parameter.

The stability region of the slow system \( A_\varepsilon(x^0,z^0) = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : \varphi(t,x^0,z^0) \rightarrow (x,z) \text{ as } t \rightarrow \infty \} \) is a \( n \)-dimensional subset of the constraint manifold \( \Gamma = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : g(x,z)=0 \} \) while the stability region \( A_{\varepsilon}(x^0,z^0) \) of the original system is a \( n+m \)-dimensional subset of \( \mathbb{R}^{n+m} \). As a consequence, the stability region of the slow system is not an approximation of the stability region of the system \((\Sigma_{\varepsilon})\) for small \( \varepsilon \). A relationship between the stability region of the TTS system \((\Sigma_{\varepsilon})\) and the stability regions of its two simplified systems, the slow and fast systems, will be investigated. The topological boundaries of these sets, simply called stability boundaries, will be respectively denoted by \( \partial A_{\varepsilon}(x^0,z^0) \), \( \partial A_{\varepsilon}(x^0,z^0) \), and \( \partial A_F(x^0,z^0) \). It will be shown that local approximations of the stability boundary of the TTS system can be obtained in terms of the stability boundaries of the slow and fast systems.

The constraint manifold \( \Gamma \) plays an essential role in the establishment of a relationship between the stability region of the TTS system \((\Sigma_{\varepsilon})\) and those of the simplified subsystems \((\Sigma_0)\) and \((\Pi_\varepsilon(x))\). The constraint manifold \( \Gamma \) is a set of equilibriums of the fast system \((\Pi_\varepsilon)\) and \( E=\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m : f(x,z)=0,g(x,z)=0 \} \), the set of equilibriums of the TTS system \((\Sigma_{\varepsilon})\), is a subset of \( \Gamma \). Typically, set \( \Gamma \) is a smooth manifold composed of several disjoint connected components (Venkatasubramanian et al., 1995). Let \( NH \subset \Gamma \) be the set of nonhyperbolic points on \( \Gamma \), that is, the points of \( \Gamma \) where \( Dg \) has at least one eigenvalue on the imaginary axis. Set \( NH \) is a manifold of dimension \( n-1 \) that separates each of the components of \( \Gamma \) into smaller connected components \( \Gamma_i \), such that \( \bigcup_i \Gamma_i = \Gamma \) (Venkatasubramanian et al., 1995).

In each connected component \( \Gamma_i \), the number of eigenvalues of \( Dg \) on the right-half of the complex plane is constant (Hill & Mareels, 1990). The connected set \( \Gamma_i \) is said to be a type-\( k \) component of \( \Gamma \) if the matrix \( Dg \), evaluated at every point of \( \Gamma_i \), has exactly \( k \) eigenvalues lying in the right-half complex plane. If all the eigenvalues of \( Dg \) calculated at points of \( \Gamma_i \) have a negative real part, then we call \( \Gamma_i \) a stable connected component of \( \Gamma \). Note that if \((x^*,z^*)\) lies on a type-\( k \) component \( \Gamma_i \) of \( \Gamma \), then \( z^* \) is a type-\( k \) hyperbolic equilibrium point of \((\Pi_\varepsilon(x^*))\).

### 3 Stability Region Characterizations

Considering the following assumptions concerning the TTS system \((\Sigma_{\varepsilon})\):

(A1) all the equilibriums are hyperbolic,

(A2) every bounded trajectory converges to an equilibrium point, and every trajectory on the stability boundary is bounded,

(A3) the stable and unstable manifolds of the equilibrium points on the stability boundary satisfy the transversality condition.

A fundamental theorem concerning the characterization of the stability boundary of TTS systems is presented below.

#### Theorem 1: (Stability Boundary of Two-Time Scale Systems)

For the TTS system \((\Sigma_{\varepsilon})\) satisfying assumptions (A1) and (A2) for sufficiently small \( \varepsilon \), let \((x_i,z_i), i=1,2,...\) be the equilibrium points on the stability boundary \( \partial A_{\varepsilon}(x^0,z^0) \) of the asymptotically stable equilibrium point \((x_i,z_i)\), for a fixed sufficiently small \( \varepsilon \), then \( \partial A_{\varepsilon}(x^0,z^0) \subseteq \bigcup_i W^s_{\varepsilon}(x_i,z_i) \). If in addition, system \((\Sigma_{\varepsilon})\) satisfies assumption (A3), then \( \partial A_{\varepsilon}(x^0,z^0) \subseteq \bigcup_i W^s_{\varepsilon}(x_i,z_i) \).
The proof of Theorem 1 can be accomplished by observing, for each fixed small ε, that all assumptions of theorem 4.1 of (Chiang et al., 1988) are satisfied. This theorem gives a complete characterization of stability boundary of TTS systems for each fixed small ε. It asserts, under assumptions (A1)-(A3), that, for each fixed small ε, the stability boundary is completely characterized as the union of the closure of stable manifolds of all equilibrium points on the stability boundary. Assumptions (A1) and (A3) are generic properties of dynamical systems in the form of (1) and, in practice, they do not need to be verified. On the contrary, assumption (A2) is not generic and requires verification. A sufficient condition for the satisfaction of (A2) is the existence of an energy function (Chiang et al., 1988).

4 Decomposition of Stability Boundary

Exploring the two-time-scale properties, we will investigate how the stability region and stability boundary of slow and fast systems can provide an approximation to the stability boundary of the TTS system.

Type-one equilibrium points play a crucial role in the stability boundary characterization of TTS systems. We next establish the relationship among type-one equilibrium points that lie on the stability boundary of the TTS system (Σ) and those that lie on the stability boundaries of the slow (Σs) and fast systems (Πε(ε)).

Theorem 2: (Stability Boundary and Type-One Equilibriums). Consider the TTS system (Σ) satisfying assumptions (A1)-(A2) for sufficiently small ε and the associated slow system (Σs) satisfying assumption (A1). Suppose (x,s,z) is a hyperbolic asymptotically stable equilibrium point of the slow system (Σs) on the stable component Γs and (x,s,z) is a type-one equilibrium point of (Σ). Then there exists an ε>0 such that the following results hold for all ε ∈ (0,ε*):

(i) (x,s,z) is a hyperbolic asymptotically stable equilibrium point of the TTS system (Σ);
(ii) the unstable equilibrium point (UEP) (x,s,z) must belong to either a stable or a type-one component of Γ. If (x,s,z) lies on a stable component, then (x,s,z) is a type-one equilibrium point of the slow system (Σs). Otherwise, (x,s,z) is a type-one equilibrium point of the fast system (Πε(ε)).
(iii) If (x,s,z) lies on the stability boundary ∂As(x,s,z) of the slow system on Γs, then (x,s,z) lies on the stability boundary ∂As(x,s,z) of the TTS system (Σ).
(iv) If (x,s,z) lies on the stability boundary ∂As(x,s,z) of the fast system (Πε(ε)) and lies on the type-one component Γo of Γ and (x,s,z) lies in the stability region Aε(x,s,z) of the slow system (Σs), then (x,s,z) lies on the stability boundary ∂As(x,s,z) of the TTS system (Σ).

Proof: The proof of (i) and (ii) is based on the fact that type-j equilibrium points of (Σ) lying on type-k components of Γ are type j+k equilibrium points of (Σ) for sufficiently small ε. This fact can be proved in two steps: (i) observing that an eigenvalue μ of fast system (Πε) is an eigenvalue of a matrix that can be regarded as a perturbation of matrix Dg and (ii) observing that an eigenvalue λ of system (Σ) is an eigenvalue of a matrix that can be regarded as a perturbation of matrix Jg=DFDg+DgTg. Then the result follows from complex variable theory (Kato, 1966). Proof of (iii) follows from Fact 2 of (Zou et al, 2003).

To prove (iv), we show the existence of points arbitrarily close to (x,s,z), such that trajectories of (Σ) starting from these points tend to the asymptotically stable equilibrium point (x,s,z) as t→∞. By hypothesis, (x,s,z) is a stable manifold of all equilibrium points on the stability boundary. Assumptions (A1) and (A3) are generic properties of dynamical systems in the form of (1) and, in practice, they do not need to be verified. On the contrary, assumption (A2) is not generic and requires verification. A sufficient condition for the satisfaction of (A2) is the existence of an energy function (Chiang et al., 1988).
left side, \((x_n, x^2)\) belongs to the stability region of the slow system and \((x_n, x^2)\) lies on the stability boundary of the fast system, as a consequence, on the right side, the equilibrium \((x_n, x^2)\) lies on the stability boundary of the TTS system for sufficiently small \(\varepsilon\).

Theorem 2 asserts that type-one equilibrium points of the TTS system must lie either on a stable or on a type-one component of \(\Gamma\). In addition, it asserts that the task of checking whether a type-one equilibrium point is on the boundary of the TTS system can be decomposed into the tasks of checking whether the type-one equilibrium point is on the boundary of the fast and/or slow systems. Result (iii) offers a scheme to check whether a type-one equilibrium point on the stable component \(\Gamma_s\) lies on the stability boundary of a TTS system by checking whether the same equilibrium is on the stability boundary of the slow system. On the other hand, result (iv) provides a scheme to check whether a type-one unstable equilibrium point on a type-one component \(\Gamma_u\) of \(\Gamma\) belongs to the stability boundary of \(\Sigma\), for sufficiently small \(\varepsilon\). Fig. 1 illustrates conclusion (iv) of Theorem 2.

Example 1: This example illustrates the main implications of Theorem 2. The following TSS nonlinear dynamical system appears in the literature of power systems (Jing et al., 2003):

\[
\begin{align*}
\dot{\omega} &= -\frac{D}{\varepsilon} \omega - \frac{1}{M} f(\delta, \theta) \\
\dot{\delta} &= -\frac{D}{\varepsilon} \delta + f(\delta, \theta) + \omega \\
\dot{\theta} &= -g(\delta, \theta)
\end{align*}
\]  

where \(f(\delta, \theta) = B_1 \theta \sin \delta - P_1\) and \(g(\delta, \theta) = \frac{1}{\varepsilon} \left(0 - B_2 \theta \cos \delta - B_2 \theta^2\right)\). The constraint set \(\Gamma = \{ (\omega, \delta, \theta) : g(\delta, \theta) = 0 \}\), shown in Fig. 2, is a two-dimensional manifold composed of two components, the stable component \(\Gamma_s\) and the type-one component \(\Gamma_u\), which are separated by the set of singular points NH. For \(M_s = 20, D_s = 0.4, D_s = 50, P_s = 4, Q_s = 0.5, B_{1s} = 10, B_{2s} = 10\), the slow system \((\Sigma_s)\) possesses a hyperbolic asymptotically SEP \((\omega_s, \delta_s, \theta_s) = (0.0, 0.5, 2, 0.81)\) on \(\Gamma_s\). As a consequence of result (i) of Theorem 2, \((\omega_s, \delta_s, \theta_s)\) is a hyperbolic asymptotically SEP of TTS system \((\Sigma)\) for sufficiently small \(\varepsilon\).

The stability boundary of the slow system is composed of the stable manifold of a type-one UEP \((\omega_s, \delta_s, \theta_s) = (0.0, 0.93, 0.5)\), which lies on the stability boundary of the slow system, and a piece of singular points (see Fig. 2). Applying result (iii) of Theorem 2, it follows that \((\omega_s, \delta_s, \theta_s)\) also lies on the stability boundary of the TTS system \((\Sigma)\) for sufficiently small \(\varepsilon\). The result (iii) is numerically confirmed in this example by checking the nonempty intersection between stability region of the TTS system and the unstable manifold of \((\omega_s, \delta_s, \theta_s)\).

Consider the same TTS system \((\Sigma)\) with the same parameters except for \(P_s = 0.5\). The constraint manifold \(\Gamma\) does not change with changes in \(P_s\), however the equilibria change. The slow system \((\Sigma_s)\) still has a hyperbolic asymptotically SEP \((\omega_s, \delta_s, \theta_s) = (0.0, 0.053, 0.946)\) on \(\Gamma_s\). Again, applying result (i) of Theorem 2, \((\omega_s, \delta_s, \theta_s)\) is a hyperbolic asymptotically SEP of TTS system \((\Sigma)\) for sufficiently small \(\varepsilon\). The type-one UEP \((\omega_s, \delta_s, \theta_s) = (0.0, 0.73, 0.075)\) of the TTS system is on the stability boundary of the fast system \((\Sigma_f)\) for sufficiently small \(\varepsilon\). The SEP of the fast system \((\omega_s, \delta_s, \theta_s)\) lies inside the stability region of the slow system (see Fig. 3) and, as a consequence of result (iv) of Theorem 2, the UEP \((\omega_s, \delta_s, \theta_s)\) lies on the stability boundary of the TTS system for sufficiently small \(\varepsilon\). This result is also confirmed in the numerical example: observing a non-empty intersection between the stability region of the TTS system and the unstable manifold of \((\omega_s, \delta_s, \theta_s)\).

We next establish the relationship between the stability boundary of the TTS system and the stability...
boundary and stability region of the fast and slow systems. Theorem 3 offers an approximation of the stability boundary of the TTS system, for sufficiently small \( \varepsilon \), by means of the union of stability regions of the fast system and the stability boundary of the slow system.

**Theorem 3: (Closeness of Stability Boundary of Slow and TTS System)** Consider system \((\Sigma_\alpha)\) satisfying assumptions (A1)-(A3), for sufficiently small \( \varepsilon \), and the associated slow system \((\Sigma_{\alpha,0})\) satisfying assumption (A1). Let \((x_{0,0},z_{0})\) be a hyperbolic asymptotically stable equilibrium point and \((x_{0},z_{0})\) be a type-one equilibrium point lying on the stability boundary \(\partial A_{\alpha}(x_{0},z_{0})\) of the slow system \((\Sigma_{\alpha,0})\). Let \(S \subset W^s_{\alpha,0}(x_{0},z_{0})\) be a connected compact subset of dimension n-1 of the stability boundary of the slow system \((\Sigma_{\alpha,0})\) containing \((x_{0},z_{0})\). Let \(N\) be a neighborhood of \(S\) in \(R^{n+m}\) and, for each \((\tilde{x},\tilde{z})\in S\), let \(F_{\tilde{x}}=W^s_{\alpha,0}(\tilde{x},\tilde{z})\cap N\), the intersection of the stability region of the fast system \((\Pi_{\alpha,0}(\tilde{x}))\) with \(N\). Then, given \(\eta>0\), there exists \(\varepsilon>0\) such that \(\bigcup_{(\tilde{x},\tilde{z})\in S} F_{\tilde{x}}\) is a (n+m)-dimensional set that is \(\varepsilon\)-close to the stability boundary \(\partial A_{\alpha}(x_{0},z_{0})\cap N\) of the TTS system for all \(\varepsilon \in (0,\varepsilon_0)\) (see Fig. 4). (Remark: A set \(A\) is \(\eta\)-close to a set \(B\) if \(d(A,B)\leq \eta\) in which \(d(A,B)=\sup_{x,y\in A,B} d(x,y)\).

**Proof:** Set \(S\) is a compact set composed of equilibrium points of the fast system \((\Pi_{\alpha})\). Consider the extended system:

\[
(\Pi_{\alpha}\times 0)\left\{\begin{array}{l}
\frac{dx}{dt}=\varepsilon f(x,z), \\
\frac{dz}{dt}=g(x,z), \\
\frac{d\varepsilon}{dt}=0.
\end{array}\right.
\]

For each \((\tilde{x},\tilde{z})\in S\), let \(E^c_{\alpha,0}(\tilde{x},\tilde{z})\) and \(E^{s}_{\alpha,0}(\tilde{x},\tilde{z})\) denote the invariant subspaces of \(R^{n+m}\times(-\varepsilon,\varepsilon)\) associated with the eigenvalues of the Jacobian of the system \((\Pi_{\alpha}\times 0)\) calculated at \((\tilde{x},\tilde{z},0)\). According to Theorem 9.1 of (Fenichel, 1979), there exists a center-stable manifold \(C^r\) for \((\Pi_{\alpha}\times 0)\) near \(S\) for sufficiently small \(\varepsilon\). This center-stable manifold \(C^r\) satisfies the following properties: (i) \(K\times\{\varepsilon=0\}\subset C^r\); (ii) \(C^r\) is locally invariant under the flow \((\Sigma_{\alpha})\times 0\); (iii) \(C^r\) is tangent to \(E_{\varepsilon}(x)\times E_{\varepsilon}(z)\) at \((\tilde{x},\tilde{z},0)\) for all \((\tilde{x},\tilde{z})\in S\). Let \((\tilde{x},\tilde{z})\) be a point lying in \(F_{\tilde{x}}\) and, as a consequence, lying in \(W^s_{\alpha,0}(\tilde{x},\tilde{z})\). Given \(\rho>0\), there exist \(\varepsilon>0\) such that \(\rho/2<\varepsilon\leq \varepsilon_0\).

Let \(P\) be the projection onto \(E_{\varepsilon}(x)\). Define \(q=P(\Phi_{\alpha}(\tilde{x},\tilde{z})-\tilde{x},\tilde{z})\). According to Chang, (1969), given \(\rho>0\), there exists a unique solution \(q(\tilde{t})\), bounded for \(\tilde{t}>0\), such that \(P(\Phi_{\alpha}(\tilde{t},\tilde{t})\tilde{t}=q\) and \(|q(\tilde{t})-\tilde{q}(\tilde{t},\tilde{t})|\leq \rho\) for every \(\tilde{t}>0\) and \(\varepsilon\) sufficiently small. Since \(q(\tilde{t})\) is bounded for \(\tilde{t}>0\) and \(q(\tilde{t},\tilde{t})\underset{\tilde{t}\to\infty}{\to} (x_{0},z_{0})\) as \(\varepsilon\to 0\), then \(\tilde{q}(\tilde{t})\to (x_{0},z_{0})\) as \(\tilde{t}\to\infty\) as a consequence of assumptions (A1) and (A2). As a consequence, \(\Phi_{\alpha}(0)\in C^r\) and \(\Phi_{\alpha}(0)\in W^s_{\alpha,0}(x_{0},z_{0})\). Since \((\tilde{x},\tilde{z})\) is a hyperbolic equilibrium point of the fast system \((\Pi_{\alpha}(\tilde{x}))\), the stable manifold \(W^s_{\alpha,0}(\tilde{x},\tilde{z})\) is also tangent to \(E_{\varepsilon}(x)\) at \((\tilde{x},\tilde{z})\). As a consequence, given a number \(\alpha>0\), there exist \(\rho>0\) such that \(\|\Pi-P(x,z)\|=\|\Phi_{\alpha}(0)\|=\|\Phi_{\alpha}(0)\times\|<\alpha\rho\). As a consequence

\[
\left|\begin{array}{c}
|\Phi_{\alpha}(\tilde{x},\tilde{z})-\tilde{q}(\tilde{t})|=

\end{array}\right|<\alpha\rho.
\]

Finally, we apply the regular perturbation technique to the system \((\Pi_{\alpha})\). Given \(\eta>0\) there exist \(\gamma>0\) such that \(\Phi_{\alpha}(\tilde{t},x,z)\in B_{\rho}(\tilde{x},\tilde{z})\) for every \(\tilde{t}>0\) such that \(\rho/2<\varepsilon<\varepsilon_0\) and \(\eta\leq \varepsilon_0\). Choosing \(\varepsilon<\varepsilon_0\), for \(\eta<\varepsilon_0\), one guarantees that \(\Phi_{\alpha}(0)\in B_{\rho}(\tilde{x},\tilde{z})\). As a consequence, there exists a point \(\tilde{t}=\Phi_{\alpha}(\tilde{x},\tilde{z})\in W^s_{\alpha,0}(x_{0},z_{0})\) that is \(\eta\)-close to \((\tilde{x},\tilde{z})\). As a consequence of Theorem 2, \((x_{0},z_{0})\in \partial A_{\alpha}(x_{0},z_{0})\) for all \(\varepsilon \in (0,\varepsilon_0)\), then \((\tilde{x},\tilde{z})\) is \(\eta\)-close to \(\partial A_{\alpha}(x_{0},z_{0})\) for all \(\varepsilon \in (0,\varepsilon_0)\). The proof is completed observing this is true for every \((\tilde{x},\tilde{z})\) in the compact set \(\bigcup_{(\tilde{x},\tilde{z})\in S} F_{\tilde{x}}\).

**Example 2:** Consider the TTS system example 1. Figure 5 shows a subset of the stability region of the slow system, and the union of sets \(F_{\tilde{x}}\) provides an approximation of the stability boundary of the TTS system.
Next theorem offers an approximation of the stability boundary of the TTS system \( \partial A^\varepsilon(x,\hat{z}) \) of \( (\Sigma) \), for sufficiently small \( \varepsilon \), by means of the union of stability boundaries of the fast systems and the stability region of the slow system.

**Theorem 4:** (Closeness of Stability Boundary of Fast Subsystem and TTS System) Consider the TTS system \( (\Sigma) \) satisfying assumption (A1)-(A3), for sufficiently small \( \varepsilon \), and the associated slow system \( (\Sigma_o) \) satisfying assumption (A1). Let \( (x,\hat{z}) \) be a hyperbolic asymptotically stable equilibrium point and \( (x_o,\hat{z}_o) \) a type-one equilibrium point on the type-one component \( \Gamma_o \) of \( \Gamma \) of the TTS system \( (\Sigma) \) lying on the stability boundary \( \partial A^\varepsilon(x,\hat{z}) \) of the TTS system \( (\Sigma) \) for sufficiently small \( \varepsilon \). Let \( Q \subset A^\varepsilon(x_o,\hat{z}_o) \) be a connected compact subset of dimension \( n \) of the stability region \( A^\varepsilon(x_o,\hat{z}_o) \) of the slow system on \( \Gamma_o \). Let \( N \) be a neighborhood of \( Q \) in \( R^{n+m} \) and for each \( (\hat{x},\hat{z}) \in Q \), define the \( (m-1) \)-dimensional set \( G_i = W^r_{\gamma_i}(\hat{x},\hat{z}) \cap N \). Then, given \( \eta>0 \), there exists \( \varepsilon^* \) such that \( \bigcup_{(\hat{x},\hat{z})\in Q} G_i \) is a \( (n+m-1) \)-dimensional set that is \( \eta \)-close to the stability boundary \( \partial A^\varepsilon(x_o,\hat{z}_o) \cap N \) of the TTS system for all \( \varepsilon \in (0,\varepsilon^*) \).

The proof can be derived using arguments very similar to those used in the proof of Theorem 3; it will be omitted as a matter of lack of space.

**4 Conclusions**

We have developed a comprehensive characterization of the stability boundary of autonomous TTS dynamical systems and explored the two-time-scale property to derive its relationship with the stability boundaries and stability regions of slow and fast systems. The derived characterization shows that subsets of the stability region of the TTS system can be sufficiently approximated by subsets of the stability boundary and stability region of slow and fast systems.

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**Referências Bibliográficas**


