ADAPTIVE SYNCHRONIZATION OF UNIFIED CHAOTIC SYSTEMS IN THE PRESENCE OF UNKNOWN PARAMETER AND DISTURBANCES

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Abstract—This paper presents an adaptive algorithm to synchronize unified chaotic systems in the presence of unknown system parameter and bounded disturbances. Based on Lyapunov-like analysis, an adaptive scheme is proposed to make the synchronization error asymptotically null. Simulation results are provided to demonstrate the effectiveness and feasibility of the proposed synchronization method.

Keywords—Synchronization, Lyapunov methods, chaotic systems.

1 Introduction

In recent years, chaos synchronization has been applied in several areas such as in electrical engineering (Xiao, 2009; Lu and Tse, 2000), biological systems (Arecchi et al., 2003), chemical engineering (Li et al., 2003) and secure communications (Lu et al., 2002; Sun et al., 2008).

Several methods have been proposed to achieve synchronization. For instance, nonlinear active control (Wang et al., 2009a; 2009b; Wang, 2010; Chen et al., 2009; Yan, 2005; Vincent, 2008; Yassen, 2008), adaptive synchronization (Al-Sawada et al., 2010; Yang, 2011; Li, 2010; Ye and Deng, 2012), backstepping design (Zhang et al., 2005; Njah and Sunday, 2009; Peng and Chen, 2008), and sliding mode control (Jang, 2002; Liu et al., 2009; Chen et al., 2007).

However, in most of the above works, it is assumed that the slave and master systems are perfectly or structurally known, i.e., the presence of unknown disturbances is not considered. For instance, the techniques in Al-Sawada et al. (2010), Yang (2011), Li (2010), Ye and Deng (2012), Dou et al. (2009), and Mahmoud (2012) achieve asymptotic convergence of the synchronization error to zero under the crucial assumption of the inexistence of disturbances. Although the assumption of free disturbance may be interesting from the theoretical point of view, from the practical perspective it is a restrictive assumption since the presence of disturbances are, in general, unavoidable. Typical disturbances include state-dependent and time-dependent functions, which can be introduced, for instance, by unexpected changes in the system dynamic due faults, changes in operation conditions, aging of equipment, and so on.

On the other hand, it is well-known that adaptive laws designed for the disturbance or modeling error free case may suffer from parameter drift. In fact, this lack of robustness in adaptive systems in the presence of unmodeled dynamics or bounded disturbances was reported in the early 1980s. Several robust modifications to counteract this have been proposed since then (Ioannou and Sun, 1995).

Motivated by the previous facts, in this paper we propose an adaptive synchronization method for unified chaotic systems in the presence of uncertain system parameter and bounded disturbances. Based on Lyapunov-like analysis, the proposed controller ensures the convergence of the synchronization error to zero, even in the presence of uncertain system parameter and bounded disturbances.

2 Problem Formulation

Consider the problem of control unified chaotic systems described by the following differential equation

\[ \dot{x}_s = A(\beta)x_s + f_s(x_s) + d_s(x_s, t) + u \]  

(1)

where \( x_s \in \mathbb{R}^3 \) is the state of the slave system, \( u \in \mathbb{R}^3 \) is the control input, \( f_s(\cdot) \) is a known map,


d_s(\cdot) is an unknown disturbance and \( \beta \) is an unknown parameter,

\[
A(\beta) = \begin{bmatrix} -25\beta - 10 & 25\beta + 10 & 0 \\ 28 - 35\beta & 29\beta - 1 & 0 \\ 0 & 0 & -\frac{8 + \beta}{3} \end{bmatrix}
\]  

(2)

and

\[
f_s(x_s) = \begin{bmatrix} 0 \\ -x_{s1}x_{s3} \\ x_{s1}x_{s2} \end{bmatrix}
\]  

(3)

We assume that the following can be established.

**Assumption 1:** On the region \( \mathbb{R}^3 \times [0, \infty) \)

\[
\|d_s(x,t)\| \leq d_{s0}
\]  

(4)

where \( d_{s0} \) is a positive constant, such that \( d_{s0} < \bar{d}_s \)

and \( \bar{d}_s \) is a known constant.

**Assumption 2:** The parameter \( \beta \) is upper bounded by a known positive constant \( \bar{\beta} \), such that \( \bar{\beta} > \beta \).

**Remark 1:** Assumption 1 is quite natural since unified chaotic systems are bounded by definition.

**Remark 2:** In case that \( \beta=0, \bar{\beta}=0.8, \) and \( \beta=1, \) system (1) becomes the Lorenz, Lü, and Chen systems, respectively, when perturbations are not present.

In order to have a well-posed problem, without loss of generality, we consider the master system as

\[
\dot{x}_m = A(\alpha)x_m + f_m(x_m) + d_m(t, x_m)
\]  

(5)

where \( x_m \in \mathbb{R}^3, \alpha \) is a known parameter and \( d_m(\cdot) \) is an unknown disturbance.

\[
A(\alpha) = \begin{bmatrix} -25\alpha - 10 & 25\alpha + 10 & 0 \\ 28 - 35\alpha & 29\alpha - 1 & 0 \\ 0 & 0 & -\frac{8 + \alpha}{3} \end{bmatrix}
\]  

(6)

and

\[
f_m(x_m) = \begin{bmatrix} 0 \\ -x_{m1}x_{m3} \\ x_{m1}x_{m2} \end{bmatrix}
\]  

(7)

**Assumption 3:** On the region \( \mathbb{R}^3 \times [0, \infty) \)

\[
\|d_m(x,t)\| \leq d_{m0}
\]  

(8)

where \( d_{m0} \) is a positive constant, such that \( d_{m0} < \bar{d}_m \) and \( \bar{d}_m \) is a known constant.

Hence, our aim is to design a feedback control \( u \), such that the state \( x_s \) of the slave chaotic system (1) tracks the state \( x_m \) of the master system (5).

Define the synchronization error \( e(t) = x_s - x_m \).

Then, from (1) and (5), we obtain the synchronization error equation

\[
\dot{e} = A(\alpha)e + \beta Bx_s + \alpha Cx_s + f_s - f_m + d + u
\]  

(9)

where \( d = d_s - d_m \), \( C = -B \) and

\[
B = \begin{bmatrix} -25 & 25 & 0 \\ -35 & 29 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}
\]  

(10)

**Remark 3:** It should be noted that in our formulation, for sake of simplicity, it was considered that \( f_m(\cdot) = f_s(\cdot) \). However, these nonlinear mappings can be unrelated, for instance, to include a priori knowledge of the disturbances.

### 3 Adaptive Synchronization

In this section, we considered the problem of asymptotic adaptive synchronization in the presence of unknown parameter and bounded disturbances. It is shown using Lyapunov-like analysis that the synchronization error converges asymptotically to zero.

**Theorem 1:** Consider the slave (1) and master (5) chaotic systems, which satisfy Assumptions 1-3, the control law

\[
u_r = -\hat{\beta}Bx_s + \alpha Cx_s + f_s - f_m + A(\alpha)e + Le + u_r
\]  

(11)

with

\[
u_r = \frac{ke}{\lambda_{\min}(K)\|f + \gamma_1 \exp(-\gamma_\nu t)\|}
\]  

(12)

\[
\hat{\beta} = -\gamma_\beta \|\gamma_2\|\|\beta - e^T KBx_s\|
\]  

(13)

where

\[
L^T P + PL = -Q, \quad P = P^T > 0, \quad Q > 0
\]

\[
K = P + P^T, \quad \gamma_5 = \lambda_{\min}(Q), \quad \gamma_\nu \geq 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0, \quad \gamma_\beta > 0
\]  

(14)

\[
k = \|K\|_F \|\bar{d}_o + \gamma_2 \bar{\beta}^2/2\|, \text{ and } \|K\|_F \text{ is the Frobenius norm of } K.
\]
Then, the slave and master systems synchronize, i.e., \( \lim_{t \to \infty} e(t) = 0 \).

**Proof:** Consider the Lyapunov function candidate

\[
V = e^T Pe + y_\beta^{-1} \tilde{\beta} \tilde{\beta} / 2
\]  

(15)

where \( \tilde{\beta} = \hat{\beta} - \beta \).

The time derivative of (15) results

\[
\dot{V} = e^T Pe + e^T Pe + \tilde{\beta} y_\beta^{-1} \tilde{\beta}
\]  

(16)

On the other hand, by using (11), the closed-loop synchronization error can be written as

\[
e = -Le - \tilde{\beta} B x_e - u_e + d
\]  

(17)

By evaluating (16) along the trajectories of (13) and (17), we obtain

\[
\dot{V} = -e^T (L^T P + P L) e - \tilde{\beta} e^T (P + P^T) B x_e
\]

\[
- e^T (P + P^T) u_e + e^T (P + P^T) d - \gamma_3 \|e\| \|\tilde{\beta}\| + e^T K B x_e \tilde{\beta}
\]  

(18)

Using now (14), (18), results

\[
\dot{V} = -e^T Q \epsilon - e^T K u_e - \gamma_3 \|e\| \|\tilde{\beta}\| + e^T K d
\]  

(19)

which, by using (14) and Assumption 1 and 3, implies

\[
\dot{V} \leq -\gamma_3 \|e\| + (\gamma_3 \|\tilde{\beta}\| + \|d\|) \|e\| + \|d\| \|e\| + \|d\| \|e\|
\]  

(20)

where \( d_0 = \delta_0 + d_m \).

Since \( \tilde{\beta} = \hat{\beta} - \beta \), it can be established that

\[
\tilde{\beta} \hat{\beta} = \frac{1}{2} \hat{\beta}^2 - \frac{1}{2} \hat{\beta}^2 - \frac{1}{2} \beta^2
\]  

(21)

Thus by employing Assumption 2, (12) and (21), (20) implies

\[
\dot{V} \leq -\gamma \|e\| + (\gamma_3 \|\tilde{\beta}\| + \|d\|) \|e\| + \frac{\|d\| \|e\|}{2}
\]  

(22)

Using definition (14), (22) implies that

\[
\dot{V} \leq -\gamma_3 \|e\|^2 + k \gamma_1 \exp(-\gamma_\epsilon t) \|e\| + \lambda \exp(-\gamma_\epsilon t)
\]  

(23)

Since \( V \) is bounded from below and non-increasing with time, we have

\[
\lim_{t \to \infty} \int_0^t \|e(\tau)\| d\tau \leq \frac{[V(0) - V_\infty]}{\gamma_3} + \frac{k \gamma_1}{\gamma_3} < \infty
\]  

(24)

where \( \lim_{t \to \infty} V(t) = V_\infty \). Notice that with the bounds on \( \hat{\beta} \) and \( e \), \( \|e\|^2 \) is uniformly continuous. Thus, from (17), it follows that \( \dot{e} \) is also bounded.

Hence, by applying the Barbalat’s Lemma (Khalil, 2001), we conclude that \( \lim_{t \to \infty} e(t) = 0 \).

### 4 Simulations

To illustrate the application of the proposed scheme, the Chen chaotic system was considered. However, in contrast to Al-Sawada et al. (2010), Yang (2011), Li (2010), Ye and Deng (2012), Dou et al. (2009), and Mahmoud (2012), we consider here the presence of the following disturbance

\[
d_t(x, t) = \left[ 3 \sin(t), 3 \sin(2t), \sin(t) \right]^T
\]  

(25)

Hence, to synchronize the slave system (1) and the master system (5), the adaptive laws (11)-(13) were used. The initial conditions for the master and slave systems were \( x_m = [1.5 \ 2 \ 5] \) and \( x_s = [4 \ 8 \ 3] \) respectively, in order to evaluate the performance of the proposed synchronization algorithm under adverse initial conditions.

The Chen system, i.e. \( \alpha = 1 \), was considered for the simulation. The others design parameter were chosen as \( k = 1 \), \( \hat{\beta}(0) = 0 \), \( \gamma_0 = 0.001 \), \( \gamma_1 = 0.2 \), \( \gamma_2 = 2 \), \( \gamma_3 = 0.05 \), and \( L = \text{diag}(0.1, 100, 5) \).

It should be noted that \( \hat{\beta} = 1 \) is only used for generation of state trajectory \( x_s(t) \). For the implementation of the control law (11), it is assumed that \( \hat{\beta} = 1 \) is unknown.

The performances in the synchronization are shown in Figures 1-6, where a logarithmic scale is used for the time axis to show the fast transient. We can see that the simulations confirm the theoretical results, that is, the synchronization algorithm is stable and the synchronization error is very small.
Figure 1. Performance in the synchronization of $x_{s1}(t)$.

Figure 2. Performance in the synchronization of $x_{s2}(t)$.

Figure 3. Performance in the synchronization of $x_{s3}(t)$. 
Figure 4. Synchronization error \( (x_{s1}(t) - x_{m1}(t)) \).

Figure 5. Synchronization error \( (x_{s2}(t) - x_{m2}(t)) \).

Figure 6. Synchronization error \( (x_{s3}(t) - x_{m3}(t)) \).
5 Conclusions

This paper has proposed a synchronization algorithm to unified chaotic systems that can be affected by uncertainty in the parameter and bounded disturbances. The proposed scheme guarantees the asymptotical convergence of the synchronization error to zero.

Referências Bibliográficas


