A Simple Technique to Handle Fields Singularities in the Finite Element Time Domain Method

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***Abstract*—The use of divergence-conforming edge basis functions of first order is proposed as a simple and efficient alternative to deal with singularities of electromagnetic fields in analyses employing the finite-element time-domain method (FETD). This treatment consists in applying this kind of function only to the edges of the tetrahedral elements laying on regions where a singular behavior of the electrical field is expected. The remaining elements in the computational domain are kept with the conventional Whitney edge elements, which are curl-conforming basis functions. Tests of this technique were made through the evaluation of the characteristic impedance and the resonant frequency of some transmission lines. Improved results have been obtained using this approach when compared to the conventional FETD formulation.**

***Index Terms*—Finite Element Time Domain (FETD), electromagnetic fields singularities, edge elements, curl-conforming and divergence-conforming basis functions.**

I. Introduction

Singularities of electromagnetic fields have been subject of special interest in the development of several scientific works regarding the numerical methods applied to electrical engineering [1]-[6]. These concern the phenomena associated with the abrupt field variations, in intensity and direction, occurring in the neighborhood of sharp edges and wires [7], [8]. Classical examples of electric field singularities are the sharp point and edge effects [9]. The singular behavior of the field makes it difficult to be precisely calculated and its modeling has been a challenge.

Several approaches developed in this ambit have been implemented for the finite element method [10], [11]. Some include modifications of the basis functions that are used to approximate the field in its mathematical expansion. Generally, these basis functions are related to the geometrical entities of elements, that is, tetrahedral vertices, edges and faces, and they can be constant, linear or higher order functions. The suitable choice of these is of great importance in determining flexibility and efficiency of the method, especially when dealing with singularities.

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Whitney forms are the most widely used basis functions in the finite-element time-domain method (FETD) [12]-[14]. These are curl-conforming vector basis functions which are perfectly solenoidal. In analyses involving electrodynamic phenomena they properly model the tangential continuity and the normal discontinuity of fields, necessary to correctly manipulate the boundary conditions between different material interfaces, and they eliminate spurious responses. The accuracy of results achieved when employing these conventional functions, however, is spoiled by regions where a singular behavior of the field is expected. A suitable explanation to this fact is because in small areas around the singularity, the field has a quasi-electrostatic characteristic which, in turn, is essentially irrotational [15].

Many of the approaches found in literature to improve the accuracy in these cases, deal with 2-D problems and employ scalar basis functions [11], [16] or [17]. Others apply adaptive techniques involving mesh refinements (*h* techniques), the increase of the basis functions order (*p* techniques) or hybrid procedures of both [18], [19]. Nevertheless, these are complex formulations and are difficult to implement. Further, these algorithms demand additional computational capacity in terms of memory and time processing requirements.

This work is distinguished for the treatment of singularities in 3-D analyses and for applying vector basis functions. It is a simple alternative to implement which has demonstrated to be efficient and computationally inexpensive when compared to the above mentioned methods.

The methodology consists in replacing the conventional curl-conforming edge basis functions by linear divergence-conforming edge functions associated only with the edges that lay on conductive wedges where singularities occur.

Results obtained with this technique are compared to those resulting from the application of the conventional FETD method, in terms of the impedance and the resonant frequency of five different transmission line configurations in which the problem of field singularities is present.

II. Proposed Technique

Used in the mathematical approximation of the electric field inside tetrahedral elements, the basis functions $\vec{W}_{ij}$ are related to the spatial discretization of the computational domain and are associated with the edges of the tetrahedral. The unknown coefficients v_{ij} assemble the expansion of the electric field $\vec{E}$ as

$$\vec{E} = -\sum_{ij} v_{ij} \vec{W}_{ij} \ , \quad (1)$$

with i and j equal to 1, 2, 3 or 4.

It is known from the Helmholtz's decomposition theorem that any vector field can be separated into irrotational (curl-free and divergence-conforming) and solenoidal (divergence-free and curl-conforming) vector field components. [20].

In electrodynamic phenomena, fields present a behavior that is predominantly solenoidal, but in regions where field singularities are encountered they approach the characteristics of static fields becoming irrotational.

A. Curl-conforming Basis Functions

In FETD formulations, the most commonly used basis functions employed in (1) are the Whitney edge basis functions which are expressed as

$$\vec{W}_{ij}^{\ C} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \ , \quad (2)$$

λ_i being the barycentric coordinate and $\nabla \lambda_i$ its gradient.

The edge element is a set of six curl-conforming basis functions, and has six degrees of freedom, one per edge, because the permutation of indices in (2) do not generate a new function, only changes its sign. Coefficients v_{ij}^C related to the curl-conforming functions of (2) have a ready physical interpretation: they are the voltages between the nodes of elements. Curl-conforming vector edge functions are characterized by their null divergence and they are mainly tangential to its associated edge.

In order to illustrate the solenoidal behavior of one of these functions in a tetrahedron face, Fig. 1 shows the $\vec{W}_{12}^C$ vectors. It can be seen that tangential components are constant along the edge 1-2 considered, while normal components vary linearly.

B. Divergence-conforming Basis Functions

Divergence-conforming vector edge functions are characterized by presenting null curl and are mainly normal to the associated edge. Coefficients v_{12}^D related to these functions are not the voltages on the edges and do not have an explicit physical meaning.

One edge basis function, of first order, which has an irrotational characteristic, is

$$\vec{W}_{ij}^{\ D} = \lambda_i \nabla \lambda_j + \lambda_j \nabla \lambda_i \, . \quad (3)$$

There are also six possible functions as (2) in a tetrahedron. In contrast with the curl-conforming basis functions of (2), divergence-conforming basis functions of (3) present linear variation in both tangential and normal directions. Fig. 2 shows the behavior of $\vec{W}_{12}^D$.

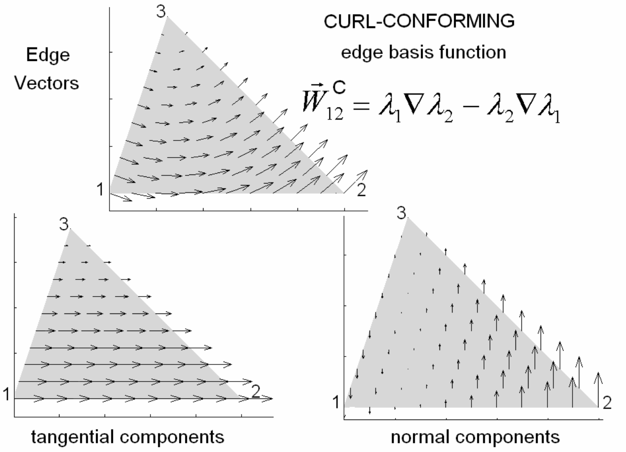

Fig. 1 – Vectors of a curl-conforming edge basis function.

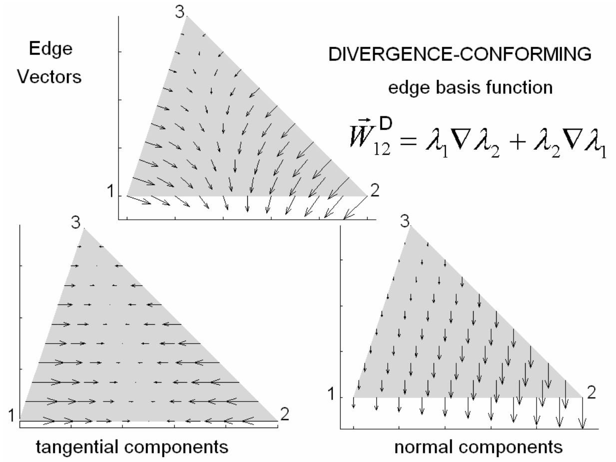

Fig. 2 – Vectors of a divergence-conforming edge basis function.

C. Formulation

The FETD formulation applied in this work is essentially the same of that found in [21]. The distinction consists in the replacement of one of the conventional basis functions of the curl-conforming kind in (2) by a divergence-conforming one in (3), only for those edges laying on sharp wedges of conductive objects, regions where a singular behavior of the electric field is predictable. The remaining elements in the computational domain are kept as the functions in (2). Thus, this method does not imply an increase in the number of degrees of freedom, because six basis functions per element are maintained. As a consequence, the time processing required for simulation is not affected when this technique is compared to the conventional FETD method.

In order to illustrate the proposed procedure, it is shown in Fig. 3 a stripline and one of its tetrahedral elements of discretization. The tetrahedron has an edge laying on the left border of the conductive strip. As well as in the conventional

FETD method, this element is also modeled with six degrees of freedom, but in this case, five of them refer to the curl-conforming functions and one to a divergence-conforming function.

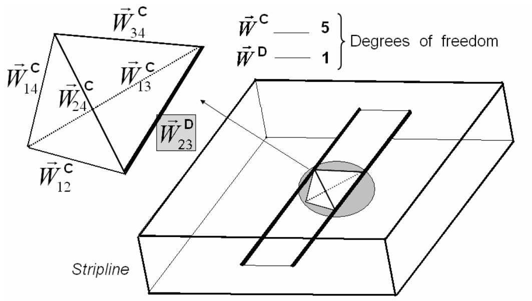

Fig. 3 – A transmission line scheme illustrating the application of the technique involving the replacement of an edge basis function.

In perfectly conductive wedges, coefficients associated to the curl-conforming edge basis functions are null, therefore, these functions do not influence the results of simulations, and so these can be eliminated and replaced by the divergence-conforming functions.

III. Numerical Experiments

The effectiveness of the proposed technique was evaluated by analyzing the characteristic impedance Z_c and the resonant frequency f_o of transmission lines in five different configurations, commonly used in microwave applications. These structures present geometric characteristics that originate field singularities, i.e. conductive borders or wedges.

Solutions obtained in simulations are compared to the theoretical values of Z_c and f_o, calculated from analytical or empirical expressions available in literature. Results are expressed in terms of the absolute value of the relative errors of each physical quantity.

Transmission lines were excited with an electric current source, having Gaussian shape in time with a peak of 1 A, and the response acquired is a voltage wave *V(t)*, used to calculate Z_c and f_o. Typically, this voltage waveform resembles that shown in Fig. 4.

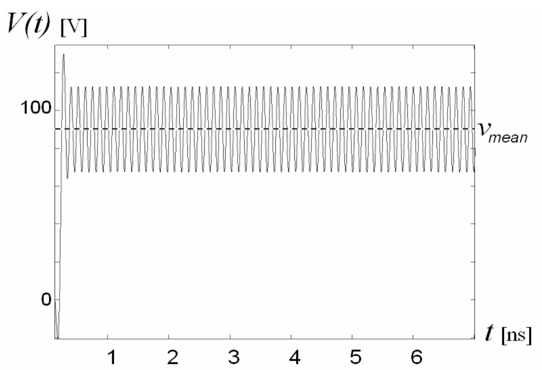

Fig. 4 – Typical voltage waveform obtained in simulations.

From the Fourier transform of *V(t)*, the resonant frequency of the transmission line is found. The characteristic impedance is obtained taking into account the mean value v_{mean} in permanent regime.

The examined structures are: a stripline with infinitesimal thickness, a pair of coupled striplines (both TEM propagation modes), a microstrip line and a stripline of finite thickness.

Tests were repeated using three meshes of different sizes.

A. *Stripline of Infinitesimal Thickness*

The simulated stripline, with infinitesimal thickness, has the dimensions represented in Fig. 5. The substrate between the perfect electric conductors (PEC) is air.

In this simulation an electrical current pulse having a width of 0.1 ns was applied in order to excite the lowest resonant frequency which is 25.0 GHz.

The characteristic impedance resulting from the simulations are compared to its theoretical value of 73.8 Ω [22].

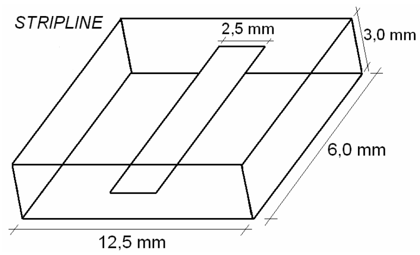

Fig. 5 – Stripline of infinitesimal thickness.

B. *Coupled Striplines*

Two coupled striplines having the same characteristics of the above mentioned one and distant 1.0 mm apart from each other are represented in Fig. 6.

Both even and odd TEM propagation modes were simulated in this line configuration. The theoretical characteristic impedance is 66.8 Ω, for the first mode, and 80.0 Ω, for the second [22]. The theoretical resonant frequency is also 25.0 GHz.

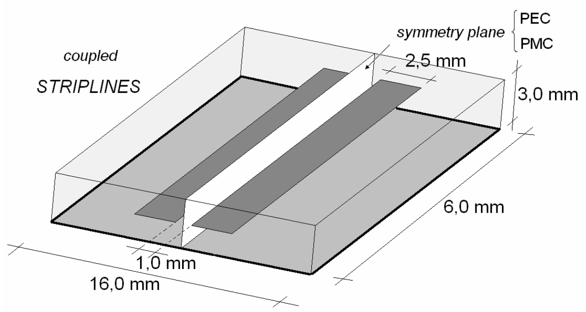

Fig. 6 – Coupled striplines. The symmetry plane is a perfect electrical conductor (PEC) for the odd TEM propagation mode, and it is a perfect magnetic conductor (PMC) for the even mode.

C. *Microstrip line*

A sketch of the simulated microstrip line is shown in Fig. 7. Alumina is the substrate with relative dielectric constant ε_r =10. The computational domain is extended above the microstrip with an air volume.

The characteristic impedance of this microstrip line is 41.1 Ω and the resonant frequency, 8.7 GHz [23].

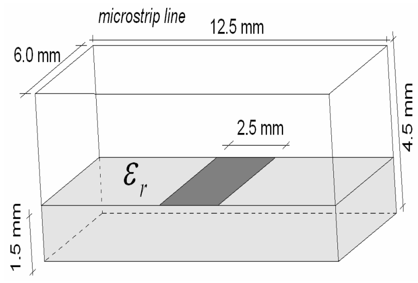

Fig. 7 – Microstrip line.

D. *Stripline of Finite Thickness*

Fig. 8 shows the sketch of the stripline of finite thickness that was simulated. Its theoretical impedance is 30.0 Ω and the resonant frequency is 25.0 GHz [24]. The transmission line is filled with air.

In this situation, the condition involving the replacement of basis functions is applied to the wedges of the inner conductor (dotted lines).

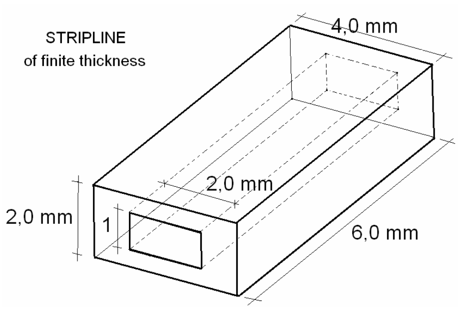

Fig. 8 – Stripline of finite thickness.

IV. Results

The results of simulations are presented in this section. The absolute value of the relative errors of each evaluated physical quantity, obtained with the application of the proposed technique and the conventional FETD method, are disposed in the form of bar graphs, for every simulated structure.

Tests were repeated employing three different meshes and results are presented in decreasing order with respect to the mean values of edge lengths adopted in simulations (1.0 mm, 0.75 mm e 0.5 mm). The transmission lines are numbered from 1 to 5 in the abscissas of graphs in accordance with Table 1.

Fig. 9 and Fig. 10 show the percentual relative errors of the characteristic impedance and the resonant frequency, respectively.

Results clearly demonstrate that the use of divergence-conforming basis functions improves the accuracy in both physical quantities.

It can be observed by comparing the two graphs, that the characteristic impedance errors are considerably greater than those of resonant frequency. This behavior can be explained by the fact that the characteristic impedance is more dependent on irrotational fields rather than the resonant frequency. Moreover, it can be noted that the errors of the resonant frequency are more affected by the inclusion of divergence-conforming basis functions. Nevertheless, the investigation of two physical quantities of distinct nature, one being predominantly of irrotational characteristic (Z_c) and other of solenoidal (f_o), has played an important role to assure the efficacy of the proposed approach.

The lack of uniformity on performances when using different meshes are due to the randomness of nonstructured meshes.

TABLE I. Tested Transmission Lines

Numbering in abscissas [a]	Configurations of tested transmission lines
1	Stripline of infinitesimal thickness
2	Coupled striplines in odd TEM propagation mode
3	Coupled striplines in even TEM propagation mode
4	Microstrip line
5	Stripline of finite thickness

[a] For reference in abscissas of graphs in Fig. 9 and 10.

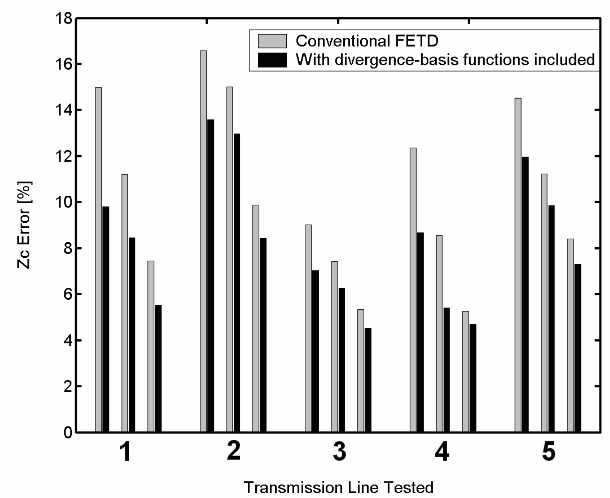

Fig. 9 – Errors of characteristic impedances of transmission lines.

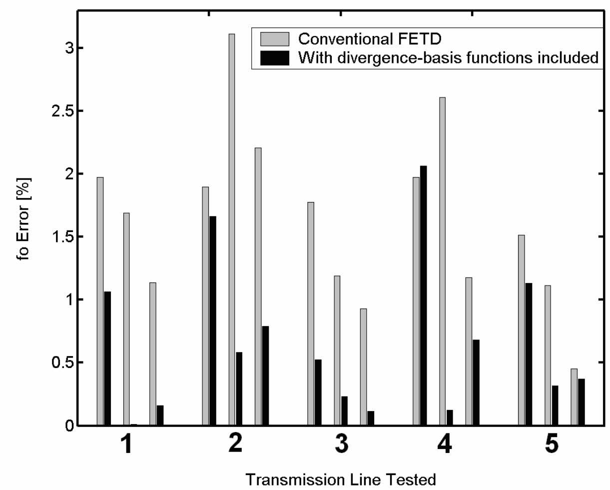

Fig. 10 – Errors of resonant frequencies of transmission lines.

V. CONCLUSIONS

A modification of the conventional FETD formulation has been proposed as an alternative method for handling field singularities. This consists on the replacement of curl-conforming by linear divergence-conforming edge basis functions associated only with the tetrahedral edges laying on conductive sharp wedges.

With this simple technique, the results involving five different configurations of transmission lines presented an accuracy improvement in the evaluation of both characteristic impedance and resonant frequency quantities.

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