14

Transmitting and Receiving Antennas

14.1 Energy Flux and Radiation Intensity

The flux of electromagnetic energy radiated from a current source at far distances is given by the time-averaged Poynting vector, calculated in terms of the radiation fields (13.10.4):

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\text{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \frac{1}{2}\left(-jk\eta\,\frac{e^{-jkr}}{4\pi r}\right)\left(jk\,\frac{e^{jkr}}{4\pi r}\right)\text{Re}\big[(\hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\phi}}F_\phi)\times(\hat{\boldsymbol{\phi}}F_\theta^* - \hat{\boldsymbol{\theta}}F_\phi^*)\big]$$

Noting that $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$, we have:

$$(\hat{\boldsymbol{\theta}}F_\theta + \hat{\boldsymbol{\phi}}F_\phi)\times(\hat{\boldsymbol{\phi}}F_\theta^* - \hat{\boldsymbol{\theta}}F_\phi^*) = \hat{\mathbf{r}}\big(|F_\theta|^2 + |F_\phi|^2\big) = \hat{\mathbf{r}}\,\big|\boldsymbol{F}_\perp(\theta,\phi)\big|^2$$

Therefore, the energy flux vector will be:

$$\boldsymbol{\mathcal{P}} = \hat{\mathbf{r}}\,\mathcal{P}_r = \hat{\mathbf{r}}\,\frac{\eta k^2}{32\pi^2 r^2}\,\big|\boldsymbol{F}_\perp(\theta,\phi)\big|^2 \tag{14.1.1}$$

Thus, the radiated energy flows radially away from the current source and attenuates with the square of the distance. The *angular distribution* of the radiated energy is described by the *radiation pattern* factor:

$$\big|\boldsymbol{F}_\perp(\theta,\phi)\big|^2 = \big|F_\theta(\theta,\phi)\big|^2 + \big|F_\phi(\theta,\phi)\big|^2 \tag{14.1.2}$$

With reference to Fig. 13.9.1, the power dP intercepting the area element $dS = r^2 d\Omega$ defines the power per unit area, or the *power density* of the radiation:

$$\boxed{\frac{dP}{dS} = \frac{dP}{r^2 d\Omega} = \mathcal{P}_r = \frac{\eta k^2}{32\pi^2 r^2}\,\big|\boldsymbol{F}_\perp(\theta,\phi)\big|^2} \qquad \text{(power density)} \tag{14.1.3}$$

The *radiation intensity* $U(\theta,\phi)$ is defined to be the power radiated per unit solid angle, that is, the quantity $dP/d\Omega = r^2 dP/dS = r^2\mathcal{P}_r$:

$$\boxed{U(\theta,\phi) = \frac{dP}{d\Omega} = r^2\mathcal{P}_r = \frac{\eta k^2}{32\pi^2}\,\big|\boldsymbol{F}_\perp(\theta,\phi)\big|^2} \qquad \text{(radiation intensity)} \tag{14.1.4}$$

The total *radiated power* is obtained by integrating Eq. (14.1.4) over all solid angles $d\Omega = \sin\theta\, d\theta d\phi$, that is, over $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$:

$$\boxed{P_{\text{rad}} = \int_0^\pi\int_0^{2\pi} U(\theta,\phi)\,d\Omega} \qquad \text{(total radiated power)} \tag{14.1.5}$$

A useful concept is that of an *isotropic radiator*—a radiator whose intensity is the same in all directions. In this case, the total radiated power P_{rad} will be equally distributed over all solid angles, that is, over the total solid angle of a sphere $\Omega_{\text{sphere}} = 4\pi$ steradians, and therefore, the *isotropic radiation intensity* will be:

$$U_I = \left(\frac{dP}{d\Omega}\right)_I = \frac{P_{\text{rad}}}{\Omega_{\text{sphere}}} = \frac{P_{\text{rad}}}{4\pi} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} U(\theta,\phi)\,d\Omega \tag{14.1.6}$$

Thus, U_I is the *average* of the radiation intensity over all solid angles. The corresponding power density of such an isotropic radiator will be:

$$\left(\frac{dP}{dS}\right)_I = \frac{U_I}{r^2} = \frac{P_{\text{rad}}}{4\pi r^2} \qquad \text{(isotropic power density)} \tag{14.1.7}$$

14.2 Directivity, Gain, and Beamwidth

The *directive gain* of an antenna system towards a given direction (θ,ϕ) is the radiation intensity normalized by the corresponding *isotropic* intensity, that is,

$$\boxed{D(\theta,\phi) = \frac{U(\theta,\phi)}{U_I} = \frac{U(\theta,\phi)}{P_{\text{rad}}/4\pi} = \frac{4\pi}{P_{\text{rad}}}\frac{dP}{d\Omega}} \qquad \text{(directive gain)} \tag{14.2.1}$$

It measures the ability of the antenna to direct its power towards a given direction. The maximum value of the directive gain, $D_{\max}$, is called the *directivity* of the antenna and will be realized towards some particular direction, say (θ_0,ϕ_0). The radiation intensity will be maximum towards that direction, $U_{\max} = U(\theta_0,\phi_0)$, so that

$$\boxed{D_{\max} = \frac{U_{\max}}{U_I}} \qquad \text{(directivity)} \tag{14.2.2}$$

The directivity is often expressed in dB,† that is, $D_{\text{dB}} = 10\log_{10} D_{\max}$. Re-expressing the radiation intensity in terms of the directive gain, we have:

$$\frac{dP}{d\Omega} = U(\theta,\phi) = D(\theta,\phi)U_I = \frac{P_{\text{rad}}D(\theta,\phi)}{4\pi} \tag{14.2.3}$$

and for the power density in the direction of (θ,ϕ):

$$\frac{dP}{dS} = \frac{dP}{r^2 d\Omega} = \frac{P_{\text{rad}}D(\theta,\phi)}{4\pi r^2} \qquad \text{(power density)} \tag{14.2.4}$$

†The term “dBi” is often used as a reminder that the directivity is with respect to the isotropic case.

Comparing with Eq. (14.1.7), we note that if the amount of power $P_{\rm rad}D(\theta,\phi)$ were emitted isotropically, then Eq. (14.2.4) would be the corresponding isotropic power density. Therefore, we will refer to $P_{\rm rad}D(\theta,\phi)$ as the *effective isotropic power*, or the *effective radiated power* (ERP) towards the (θ,ϕ)-direction.

In the direction of maximum gain, the quantity $P_{\rm rad}D_{\max}$ will be referred to as the *effective isotropic radiated power* (EIRP). It defines the maximum power density achieved by the antenna:

$$\boxed{\left(\frac{dP}{dS}\right)_{\max} = \frac{P_{\rm EIRP}}{4\pi r^2}}\,, \qquad \text{where} \quad P_{\rm EIRP} = P_{\rm rad}D_{\max} \tag{14.2.5}$$

Usually, communicating antennas—especially highly directive ones such as dish antennas—are oriented to point towards the maximum directive gain of each other.

A related concept is that of the *power gain*, or simply the *gain* of an antenna. It is defined as in Eq. (14.2.1), but instead of being normalized by the total radiated power, it is normalized to the total power P_T *accepted by the antenna terminals* from a connected transmitter, as shown in Fig. 14.2.1:

$$\boxed{G(\theta,\phi) = \frac{U(\theta,\phi)}{P_T/4\pi} = \frac{4\pi}{P_T}\frac{dP}{d\Omega}} \qquad \text{(power gain)} \tag{14.2.6}$$

We will see in Sec. 14.4 that the power P_T delivered to the antenna terminals is at most half the power produced by the generator—the other half being dissipated as heat in the generator's internal resistance.

Moreover, the power P_T may differ from the power radiated, $P_{\rm rad}$, because of several loss mechanisms, such as ohmic losses of the currents flowing on the antenna wires or losses in the dielectric surrounding the antenna.

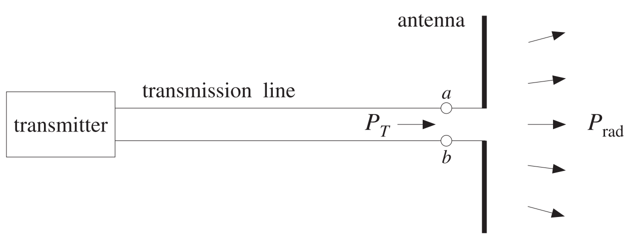

Fig. 14.2.1 Power delivered to an antenna versus power radiated.

The definition of power gain does not include any reflection losses arising from improper matching of the transmission line to the antenna input impedance [95]. The *efficiency factor* of the antenna is defined by:

$$e = \frac{P_{\rm rad}}{P_T} \qquad \Rightarrow \qquad P_{\rm rad} = eP_T \tag{14.2.7}$$

In general, $0 \le e \le 1$. For a lossless antenna the efficiency factor will be unity and $P_{\rm rad} = P_T$. In such an ideal case, there is no distinction between directive and power gain. Using Eq. (14.2.7) in (14.2.1), we find $G = 4\pi U/P_T = e4\pi U/P_{\rm rad}$, or,

$$\boxed{G(\theta,\phi) = eD(\theta,\phi)} \tag{14.2.8}$$

The *maximum gain* is related to the directivity by $G_{\max} = eD_{\max}$. It follows that the effective radiated power can be written as $P_{\rm rad}D(\theta,\phi) = P_TG(\theta,\phi)$, and the EIRP, $P_{\rm EIRP} = P_{\rm rad}D_{\max} = P_TG_{\max}$.

The angular distribution functions we defined thus far, that is, $G(\theta,\phi)$, $D(\theta,\phi)$, $U(\theta,\phi)$ are all proportional to each other. Each brings out a different aspect of the radiating system. In describing the angular distribution of radiation, it proves convenient to consider it relative to its maximal value. Thus, we define the normalized power pattern, or *normalized gain* by:

$$\boxed{g(\theta,\phi) = \frac{G(\theta,\phi)}{G_{\max}}} \qquad \text{(normalized gain)} \tag{14.2.9}$$

Because of the proportionality of the various angular functions, we have:

$$g(\theta,\phi) = \frac{G(\theta,\phi)}{G_{\max}} = \frac{D(\theta,\phi)}{D_{\max}} = \frac{U(\theta,\phi)}{U_{\max}} = \frac{|\mathbf{F}_\perp(\theta,\phi)|^2}{|\mathbf{F}_\perp|^2_{\max}} \tag{14.2.10}$$

Writing $P_TG(\theta,\phi) = P_TG_{\max}\,g(\theta,\phi)$, we have for the power density:

$$\frac{dP}{dS} = \frac{P_TG_{\max}}{4\pi r^2}\,g(\theta,\phi) = \frac{P_{\rm EIRP}}{4\pi r^2}\,g(\theta,\phi) \tag{14.2.11}$$

This form is useful for describing communicating antennas and radar. The normalized gain is usually displayed in a polar plot with polar coordinates (ρ,θ) such that $\rho = g(\theta)$, as shown in Fig. 14.2.2. (This figure depicts the gain of a half-wave dipole antenna given by $g(\theta) = \cos^2(0.5\pi\cos\theta)/\sin^2\theta$.) The 3-dB, or half-power, beamwidth is defined as the difference $\Delta\theta_B = \theta_2 - \theta_1$ of the 3-dB angles at which the normalized gain is equal to $1/2$, or, -3 dB.

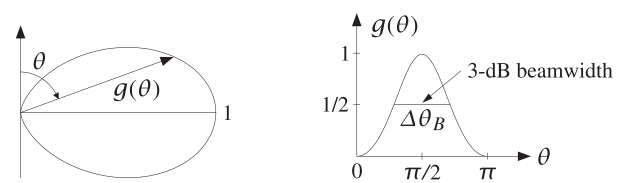

Fig. 14.2.2 Polar and regular plots of normalized gain versus angle.

The MATLAB functions `dbp`, `abp`, `dbz`, `abz` given in Appendix H allow the plotting of the gain in dB or in absolute units versus the polar angle θ or the azimuthal angle ϕ. Their typical usage is as follows:

```
dbp(theta, g, rays, Rm, width);     % polar gain plot in dB
abp(theta, g, rays, width);         % polar gain plot in absolute units
dbz(phi, g, rays, Rm, width);       % azimuthal gain plot in dB
abz(phi, g, rays, width);           % azimuthal gain plot in absolute units
```

Example 14.2.1: A TV station is transmitting 10 kW of power with a gain of 15 dB towards a particular direction. Determine the peak and rms value of the electric field E at a distance of 5 km from the station.

Solution: The gain in absolute units will be $G = 10^{G_{dB}/10} = 10^{15/10} = 31.62$. It follows that the radiated EIRP will be $P_{EIRP} = P_T G = 10 \times 31.62 = 316.2$ kW. The electric field at distance $r = 5$ km is obtained from Eq. (14.2.5):

$$\frac{dP}{dS} = \frac{P_{EIRP}}{4\pi r^2} = \frac{1}{2\eta}E^2 \quad \Rightarrow \quad E = \frac{1}{r}\sqrt{\frac{\eta P_{EIRP}}{2\pi}}$$

This gives $E = 0.87$ V/m. The rms value is $E_{rms} = E/\sqrt{2} = 0.62$ V/m. □

Another useful concept is that of the *beam solid angle* of an antenna. The definition is motivated by the case of a highly directive antenna, which concentrates all of its radiated power P_{rad} into a small solid angle $\Delta\Omega$, as illustrated in Fig. 14.2.3.

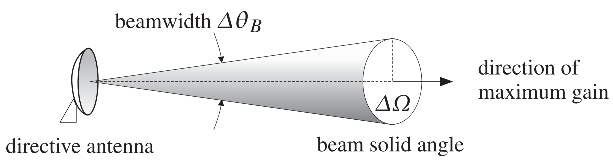

Fig. 14.2.3 Beam solid angle and beamwidth of a highly directive antenna.

The radiation intensity in the direction of the solid angle will be:

$$U = \frac{\Delta P}{\Delta\Omega} = \frac{P_{rad}}{\Delta\Omega} \tag{14.2.12}$$

where $\Delta P = P_{rad}$ by assumption. It follows that: $D_{max} = 4\pi U/P_{rad} = 4\pi/\Delta\Omega$, or,

$$\boxed{D_{max} = \frac{4\pi}{\Delta\Omega}} \tag{14.2.13}$$

Thus, the more concentrated the beam, the higher the directivity. Although (14.2.13) was derived under the assumption of a highly directive antenna, it may be used as the *definition* of the beam solid angle for any antenna, that is,

$$\boxed{\Delta\Omega = \frac{4\pi}{D_{max}}} \qquad \text{(beam solid angle)} \tag{14.2.14}$$

Using $D_{max} = U_{max}/U_I$ and Eq. (14.1.6), we have

$$\Delta\Omega = \frac{4\pi U_I}{U_{max}} = \frac{1}{U_{max}}\int_0^{\pi}\int_0^{2\pi} U(\theta,\phi)\,d\Omega\,, \qquad \text{or,}$$

$$\boxed{\Delta\Omega = \int_0^{\pi}\int_0^{2\pi} g(\theta,\phi)\,d\Omega} \qquad \text{(beam solid angle)} \tag{14.2.15}$$

where $g(\theta,\phi)$ is the normalized gain of Eq. (14.2.10). Writing $P_{rad} = 4\pi U_I$, we have:

$$\Delta\Omega = \frac{P_{rad}}{U_{max}} \qquad \Rightarrow \qquad U_{max} = \frac{P_{rad}}{\Delta\Omega} \tag{14.2.16}$$

This is the general case of Eq. (14.2.12). We can also write:

$$P_{rad} = U_{max}\Delta\Omega \tag{14.2.17}$$

This is convenient for the numerical evaluation of P_{rad}. To get a measure of the beamwidth of a highly directive antenna, we assume that the directive gain is equal to its maximum *uniformly* over the entire solid angle $\Delta\Omega$ in Fig. 14.2.3, that is, $D(\theta,\phi) = D_{max}$, for $0 \le \theta \le \Delta\theta_B/2$. This implies that the normalized gain will be:

$$g(\theta,\phi) = \begin{cases} 1, & \text{if} \quad 0 \le \theta \le \Delta\theta_B/2 \\ 0, & \text{if} \quad \Delta\theta_B/2 < \theta \le \pi \end{cases}$$

Then, it follows from the definition (14.2.15) that:

$$\Delta\Omega = \int_0^{\Delta\theta_B/2}\int_0^{2\pi} d\Omega = \int_0^{\Delta\theta_B/2}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi = 2\pi\left(1-\cos\frac{\Delta\theta_B}{2}\right) \tag{14.2.18}$$

Using the approximation $\cos x \simeq 1 - x^2/2$, we obtain for small beamwidths:

$$\boxed{\Delta\Omega = \frac{\pi}{4}(\Delta\theta_B)^2} \tag{14.2.19}$$

and therefore the directivity can be expressed in terms of the beamwidth:

$$\boxed{D_{max} = \frac{16}{\Delta\theta_B^2}} \tag{14.2.20}$$

Example 14.2.2: Find the beamwidth in degrees of a lossless dish antenna with gain of 15 dB. The directivity and gain are equal in this case, therefore, Eq. (14.2.20) can be used to calculate the beamwidth: $\Delta\theta_B = \sqrt{16/D}$, where $D = G = 10^{15/10} = 31.62$. We find $\Delta\theta_B = 0.71$ rads, or $\Delta\theta_B = 40.76^o$.

For an antenna with 40 dB gain/directivity, we would have $D = 10^4$ and find $\Delta\theta_B = 0.04$ rads $= 2.29^o$. □

Example 14.2.3: A satellite in a geosynchronous orbit of 36,000 km is required to have complete earth coverage. What is its antenna gain in dB and its beamwidth? Repeat if the satellite is required to have coverage of an area equal the size of continental US.

Solution: The radius of the earth is $R = 6400$ km. Looking down from the satellite the earth appears as a flat disk of area $\Delta S = \pi R^2$. It follows that the subtended solid angle and the corresponding directivity/gain will be:

$$\Delta\Omega = \frac{\Delta S}{r^2} = \frac{\pi R^2}{r^2} \quad \Rightarrow \quad D = \frac{4\pi}{\Delta\Omega} = \frac{4r^2}{R^2}$$

With $r = 36{,}000$ km and $R = 6400$ km, we find $D = 126.56$ and in dB, $D_{dB} = 10\log_{10}D = 21.02$ dB. The corresponding beamwidth will be $\Delta\theta_B = \sqrt{16/D} = 0.36$ rad $= 20.37^o$.

For the continental US, the coast-to-coast distance of 3000 mi, or 4800 km, translates to an area of radius $R = 2400$ km, which leads to $D = 900$ and $D_{\text{dB}} = 29.54$ dB. The beamwidth is in this case $\Delta\theta_B = 7.64^\text{o}$.

Viewing the earth as a flat disk overestimates the required angle $\Delta\theta_B$ for earth coverage. Looking down from a satellite at a height r, the angle between the vertical and the tangent to the earth's surface is given by $\sin\theta = R/(r+R)$, which gives for $r = 36{,}000$ km, $\theta = 8.68^\text{o}$. The subtended angle will be then $\Delta\theta_B = 2\theta = 0.303$ rad $= 17.36^\text{o}$. It follows that the required antenna gain should be $G = 16/\Delta\theta_B^2 = 174.22 = 22.41$ dB. The flat-disk approximation is more accurate for smaller areas on the earth's surface that lie directly under the satellite. □

Example 14.2.4: The radial distance of a geosynchronous orbit can be calculated by equating centripetal and gravitational accelerations, and requiring that the angular velocity of the satellite corresponds to the period of 1 day, that is, $\omega = 2\pi/T$, where $T = 24$ hr $= 86\,400$ sec. Let m be the mass of the satellite and $M_\oplus$ the mass of the earth (see Appendix A):

$$\frac{GmM_\oplus}{r^2} = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 r \quad\Rightarrow\quad r = \left(\frac{GM_\oplus T^2}{4\pi^2}\right)^{1/3}$$

The distance r is measured from the Earth's center. The corresponding height from the surface of the Earth is $h = r - R$. For the more precise value of $R = 6378$ km, the calculated values are:

$$r = 42\,237 \text{ km} = 26\,399 \text{ mi}$$
$$h = 35\,860 \text{ km} = 22\,414 \text{ mi}$$

14.3 Effective Area

When an antenna is operating as a receiving antenna, it extracts a certain amount of power from an incident electromagnetic wave. As shown in Fig. 14.3.1, an incident wave coming from a far distance may be thought of as a uniform plane wave being intercepted by the antenna.

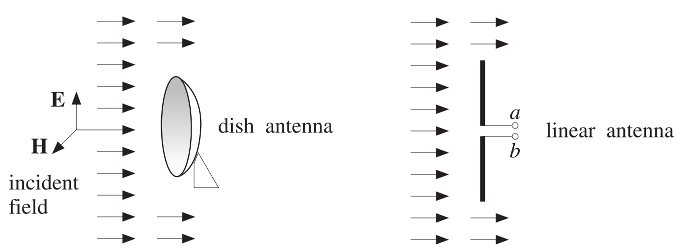

Fig. 14.3.1 Effective area of an antenna.

The incident electric field sets up currents on the antenna. Such currents may be represented by a Thévenin-equivalent generator, which delivers power to any connected receiving load impedance.

The induced currents also re-radiate an electric field (referred to as the scattered field), which interferes with the incident field causing a shadow region behind the antenna, as shown in Fig. 14.3.1.

The total electric field outside the antenna will be the sum of the incident and re-radiated fields. For a perfectly conducting antenna, the boundary conditions are that the tangential part of the total electric field vanish on the antenna surface. In Chap. 20, we apply these boundary conditions to obtain and solve Hallén's and Pocklington's integral equations satisfied by the induced current.

The power density of the incident wave at the location of the receiving antenna can be expressed in terms of the electric field of the wave, $\mathcal{P}_{\text{inc}} = E^2/2\eta$.

The *effective area* or *effective aperture* A of the antenna is defined to be that area which when intercepted by the incident power density $\mathcal{P}_{\text{inc}}$ gives the amount of *received power* P_R available at the antenna output terminals [95]:

$$\boxed{P_R = A\mathcal{P}_{\text{inc}}} \tag{14.3.1}$$

For a lossy antenna, the available power at the terminals will be somewhat less than the extracted radiated power P_{rad}, by the efficiency factor $P_R = eP_{\text{rad}}$. Thus, we may also define the *maximum effective aperture* A_m as the area which extracts the power P_{rad} from the incident wave, that is, $P_{\text{rad}} = A_m\mathcal{P}_{\text{inc}}$. It follows that:

$$A = eA_m \tag{14.3.2}$$

The effective area depends on the *direction* of arrival (θ,ϕ) of the incident wave. For all antennas, it can be shown that the effective area $A(\theta,\phi)$ is related to the power gain $G(\theta,\phi)$ and the wavelength $\lambda = c/f$ as follows:

$$\boxed{G(\theta,\phi) = \frac{4\pi A(\theta,\phi)}{\lambda^2}} \tag{14.3.3}$$

Similarly, because $G(\theta,\phi) = eD(\theta,\phi)$, the maximum effective aperture will be related to the directive gain by:

$$D(\theta,\phi) = \frac{4\pi A_m(\theta,\phi)}{\lambda^2} \tag{14.3.4}$$

In practice, the quoted effective area A of an antenna is the value corresponding to the direction of maximal gain $G_{\max}$. We write in this case:

$$\boxed{G_{\max} = \frac{4\pi A}{\lambda^2}} \tag{14.3.5}$$

Similarly, we have for the directivity $D_{\max} = 4\pi A_m/\lambda^2$. Because $D_{\max}$ is related to the beam solid angle by $D_{\max} = 4\pi/\Delta\Omega$, it follows that

$$D_{\max} = \frac{4\pi}{\Delta\Omega} = \frac{4\pi A_m}{\lambda^2} \quad\Rightarrow\quad A_m\Delta\Omega = \lambda^2 \tag{14.3.6}$$

Writing $\lambda = c/f$, we may express Eq. (14.3.5) in terms of frequency:

$$G_{\max} = \frac{4\pi f^2 A}{c^2} \tag{14.3.7}$$

The effective area is not equal to the physical area of an antenna. For example, linear antennas do not even have any characteristic physical area. For dish or horn antennas, on the other hand, the effective area is typically a fraction of the physical area (about 55–65 percent for dishes and 60–80 percent for horns.) For example, if the dish has a diameter of d meters, then we have:

$$\boxed{A = e_a \frac{1}{4}\pi d^2} \qquad \text{(effective area of dish antenna)} \tag{14.3.8}$$

where e_a is the *aperture efficiency* factor, typically $e_a = 0.55$–0.65. Combining Eqs. (14.3.5) and (14.3.8), we obtain:

$$\boxed{G_{\max} = e_a\left(\frac{\pi d}{\lambda}\right)^2} \tag{14.3.9}$$

Antennas fall into two classes: *fixed-area antennas*, such as dish antennas, for which A is independent of frequency, and *fixed-gain antennas*, such as linear antennas, for which G is independent of frequency. For fixed-area antennas, the gain *increases* quadratically with f. For fixed-gain antennas, A *decreases* quadratically with f.

Example 14.3.1: Linear antennas are fixed-gain antennas. For example, we will see in Sec. 15.1 that the gains of a (lossless) Hertzian dipole, a halfwave dipole, and a monopole antenna are the constants:

$$G_{\text{hertz}} = 1.5, \quad G_{\text{dipole}} = 1.64, \quad G_{\text{monopole}} = 3.28$$

Eq. (14.3.5) gives the effective areas $A = G\lambda^2/4\pi$:

$$A_{\text{hertz}} = 0.1194\lambda^2, \quad A_{\text{dipole}} = 0.1305\lambda^2, \quad A_{\text{monopole}} = 0.2610\lambda^2$$

In all cases the effective area is proportional to λ^2 and decreases with f^2. In the case of the commonly used monopole antenna, the effective area is approximately equal to a rectangle of sides λ and $\lambda/4$, the latter being the physical length of the monopole. □

Example 14.3.2: Determine the gain in dB of a dish antenna of diameter of 0.5 m operating at a satellite downlink frequency of 4 GHz and having 60% aperture efficiency. Repeat if the downlink frequency is 11 GHz. Repeat if the diameter is doubled to 1 m.

Solution: The effective area and gain of a dish antenna with diameter d is:

$$A = e_a\frac{1}{4}\pi d^2 \quad \Rightarrow \quad G = \frac{4\pi A}{\lambda^2} = e_a\left(\frac{\pi d}{\lambda}\right)^2 = e_a\left(\frac{\pi f d}{c}\right)^2$$

The calculated gains G in absolute and dB units are in the four cases:

	$d = 0.5$ m	$d = 1$ m
$f =$ 4 GHz	263 = 24 dB	1052 = 30 dB
$f =$ 11 GHz	1990 = 33 dB	7960 = 39 dB

Doubling the diameter (or the frequency) increases the gain by 6 dB, or a factor of 4. Conversely, if a dish antenna is to have a desired gain G (for example, to achieve a desired beamwidth), the above equation can be solved for the required diameter d in terms of G and f. □

The beamwidth of a dish antenna can be estimated by combining the approximate expression (14.2.20) with (14.3.5) and (14.3.8). Assuming a lossless antenna with diameter d and 100% aperture efficiency, and taking Eq. (14.2.20) literally, we have:

$$G_{\max} = \frac{4\pi A}{\lambda^2} = \left(\frac{\pi d}{\lambda}\right)^2 = D_{\max} = \frac{16}{\Delta\theta_B^2}$$

Solving for $\Delta\theta_B$, we obtain the expression in radians and in degrees:

$$\Delta\theta_B = \frac{4}{\pi}\frac{\lambda}{d} = 1.27\frac{\lambda}{d}, \qquad \Delta\theta_B = 73^{\text{o}}\frac{\lambda}{d} \tag{14.3.10}$$

Thus, the beamwidth depends inversely on the antenna diameter. In practice, quick estimates of the 3-dB beamwidth in degrees are obtained by replacing Eq. (14.3.10) by the formula [757]:

$$\boxed{\Delta\theta_B = 1.22\frac{\lambda}{d} = 70^{\text{o}}\frac{\lambda}{d}} \qquad \text{(3-dB beamwidth of dish antenna)} \tag{14.3.11}$$

The constant 70^{o} represents only a rough approximation (other choices are in the range 65–75°.) Solving for the ratio $d/\lambda = 1.22/\Delta\theta_B$ (here, $\Delta\theta_B$ is in radians), we may express the maximal gain inversely with $\Delta\theta_B^2$ as follows:

$$G_{\max} = e_a\left(\frac{\pi d}{\lambda}\right)^2 = \frac{e_a\pi^2(1.22)^2}{\Delta\theta_B^2}$$

For a typical aperture efficiency of 60%, this expression can be written in the following approximate form, with $\Delta\theta_B$ given in degrees:

$$\boxed{G_{\max} = \frac{30\,000}{\Delta\theta_B^2}} \tag{14.3.12}$$

Equations (14.3.11) and (14.3.12) must be viewed as approximate design guidelines, or rules of thumb [757], for the beamwidth and gain of a dish antenna.

Example 14.3.3: For the 0.5-m antenna of the previous example, estimate its beamwidth for the two downlink frequencies of 4 GHz and 11 GHz.

The operating wavelengths are in the two cases: $\lambda = 7.5$ cm and $\lambda = 2.73$ cm. Using Eq. (14.3.11), we find $\Delta\theta_B = 10.5^{\text{o}}$ and $\Delta\theta_B = 3.8^{\text{o}}$. □

Example 14.3.4: A geostationary satellite at height of 36,000 km is required to have earth coverage. Using the approximate design equations, determine the gain in dB and the diameter of the satellite antenna for a downlink frequency of 4 GHz. Repeat for 11 GHz.

Solution: This problem was considered in Example 14.2.3. The beamwidth angle for earth coverage was found to be $\Delta\theta_B = 17.36^{\text{o}}$. From Eq. (14.3.11), we find:

$$d = \lambda\frac{70^{\text{o}}}{\Delta\theta_B} = 7.5\frac{70^{\text{o}}}{17.36^{\text{o}}} = 30 \text{ cm}$$

From Eq. (14.3.12), we find:

$$G = \frac{30\,000}{\Delta\theta_B^2} = \frac{30\,000}{17.36^2} = 100 = 20 \text{ dB}$$

For 11 GHz, we find $d = 11$ cm, and G remains the same. □

In Eqs. (14.2.20) and (14.3.12), we implicitly assumed that the radiation pattern was independent of the azimuthal angle ϕ. When the pattern is not azimuthally symmetric, we may define two orthogonal polar directions parametrized, say, by angles θ_1 and θ_2, as shown in Fig. 14.3.2.

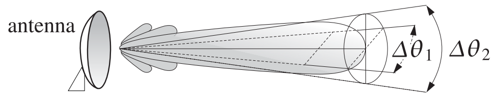

Fig. 14.3.2 Half-power beamwidths in two principal polar directions.

In this case $d\Omega = d\theta_1\, d\theta_2$ and we may approximate the beam solid angle by the product of the corresponding 3-dB beamwidths in these two directions, $\Delta\Omega = \Delta\theta_1\,\Delta\theta_2$. Then, the directivity takes the form (with the angles in radians and in degrees):

$$D_{\max} = \frac{4\pi}{\Delta\Omega} = \frac{4\pi}{\Delta\theta_1\,\Delta\theta_2} = \frac{41\,253}{\Delta\theta_1^{\circ}\,\Delta\theta_2^{\circ}} \tag{14.3.13}$$

Equations (14.3.12) and (14.3.13) are examples of a more general expression that relates directivity or gain to the 3-dB beamwidths for aperture antennas [661,673]:

$$\boxed{G_{\max} = \frac{p}{\Delta\theta_1\,\Delta\theta_2}} \tag{14.3.14}$$

where p is a *gain-beamwidth constant* whose value depends on the particular aperture antenna. We will see several examples of this relationship in Chapters 16 and 17. Practical values of p fall in the range 25 000–35 000 (with the beamwidth angles in degrees.)

14.4 Antenna Equivalent Circuits

To a generator feeding a transmitting antenna as in Fig. 14.2.1, the antenna appears as a load. Similarly, a receiver connected to a receiving antenna's output terminals will appear to the antenna as a load impedance. Such simple equivalent circuit representations of transmitting and receiving antennas are shown in Fig. 14.4.1, where in both cases V is the equivalent open-circuit Thévenin voltage.

In the transmitting antenna case, the antenna is represented by a load impedance Z_A, which in general will have both a resistive and a reactive part, $Z_A = R_A + jX_A$. The reactive part represents energy stored in the fields near the antenna, whereas the resistive part represents the power losses which arise because (a) power is radiated away from the antenna and (b) power is lost into heat in the antenna circuits and in the medium surrounding the antenna.

The generator has its own internal impedance $Z_G = R_G + jX_G$. The current at the antenna input terminals will be $I_{\rm in} = V/(Z_G + Z_A)$, which allows us to determine (a) the total power $P_{\rm tot}$ produced by the generator, (b) the power P_T delivered to the antenna terminals, and (c) the power P_G lost in the generator's internal resistance R_G. These are:

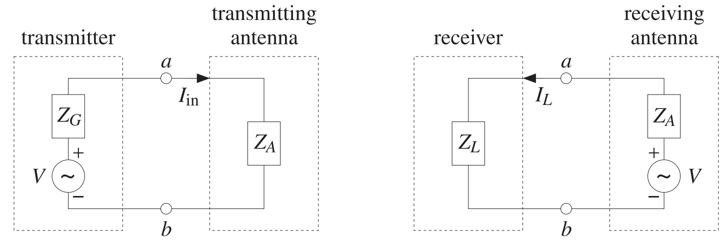

Fig. 14.4.1 Circuit equivalents of transmitting and receiving antennas.

$$\begin{gathered} P_{\rm tot} = \frac{1}{2}{\rm Re}(VI_{\rm in}^*) = \frac{1}{2}\frac{|V|^2(R_G+R_A)}{|Z_G+Z_A|^2} \\ P_T = \frac{1}{2}|I_{\rm in}|^2 R_A = \frac{1}{2}\frac{|V|^2 R_A}{|Z_G+Z_A|^2}\,, \quad P_G = \frac{1}{2}|I_{\rm in}|^2 R_G = \frac{1}{2}\frac{|V|^2 R_G}{|Z_G+Z_A|^2} \end{gathered} \tag{14.4.1}$$

It is evident that $P_{\rm tot} = P_T + P_G$. A portion of the power P_T delivered to the antenna is radiated away, say an amount $P_{\rm rad}$, and the rest is dissipated as ohmic losses, say $P_{\rm ohm}$. Thus, $P_T = P_{\rm rad} + P_{\rm ohm}$. These two parts can be represented conveniently by equivalent resistances by writing $R_A = R_{\rm rad} + R_{\rm ohm}$, where $R_{\rm rad}$ is referred to as the *radiation resistance*. Thus, we have,

$$P_T = \frac{1}{2}|I_{\rm in}|^2 R_A = \frac{1}{2}|I_{\rm in}|^2 R_{\rm rad} + \frac{1}{2}|I_{\rm in}|^2 R_{\rm ohm} = P_{\rm rad} + P_{\rm ohm} \tag{14.4.2}$$

The efficiency factor of Eq. (14.2.7) is evidently:

$$e = \frac{P_{\rm rad}}{P_T} = \frac{R_{\rm rad}}{R_A} = \frac{R_{\rm rad}}{R_{\rm rad} + R_{\rm ohm}}$$

To maximize the amount of power P_T delivered to the antenna (and thus minimize the power lost in the generator's internal resistance), the load impedance must satisfy the usual *conjugate matching* condition:

$$Z_A = Z_G^* \quad \Leftrightarrow \quad R_A = R_G\,, \quad X_A = -X_G$$

In this case, $|Z_G + Z_A|^2 = (R_G + R_A)^2 + (X_G + X_A)^2 = 4R_G^2$, and it follows that the maximum power transferred to the load will be one-half the total—the other half being lost in R_G, that is,

$$\boxed{P_{T,\max} = \frac{1}{2}P_{\rm tot} = \frac{|V|^2}{8R_G}} \tag{14.4.3}$$

In the notation of Chap. 12, this is the *available* power from the generator. If the generator and antenna are mismatched, we have:

$$P_T = \frac{|V|^2}{8R_G}\frac{4R_A R_G}{|Z_A + Z_G|^2} = P_{T,\max}\big(1 - |\Gamma_{\rm gen}|^2\big)\,, \qquad \Gamma_{\rm gen} = \frac{Z_A - Z_G^*}{Z_A + Z_G} \tag{14.4.4}$$

Eq. (14.4.3) is often written in terms of the rms value of the source, that is, $V_{\rm rms} = |V|/\sqrt{2}$, which leads to $P_{T,\max} = V_{\rm rms}^2/4R_G$.

The case of a receiving antenna is similar. The induced currents on the antenna can be represented by a Thévenin-equivalent generator (the open-circuit voltage at the antenna output terminals) and an internal impedance Z_A. A consequence of the reciprocity principle is that Z_A is the same whether the antenna is transmitting or receiving.

The current into the load is $I_L = V/(Z_A + Z_L)$, where the load impedance is $Z_L = R_L + jX_L$. As before, we can determine the total power $P_{\rm tot}$ produced by the generator (i.e., intercepted by the antenna) and the power P_R delivered to the receiving load:

$$P_{\rm tot} = \frac{1}{2}{\rm Re}(VI_L^*) = \frac{1}{2}\frac{|V|^2(R_L+R_A)}{|Z_L+Z_A|^2}\,, \quad P_R = \frac{1}{2}|I_L|^2 R_L = \frac{1}{2}\frac{|V|^2 R_L}{|Z_L+Z_A|^2} \tag{14.4.5}$$

Under conjugate matching, $Z_L = Z_A^*$, we find the maximum power delivered to the load:

$$\boxed{P_{R,\max} = \frac{|V|^2}{8R_A}} \tag{14.4.6}$$

If the load and antenna are mismatched, we have:

$$P_R = \frac{|V|^2}{8R_A}\frac{4R_AR_L}{|Z_L+Z_A|^2} = P_{R,\max}\big(1-|\Gamma_{\rm load}|^2\big)\,, \qquad \Gamma_{\rm load} = \frac{Z_L - Z_A^*}{Z_L+Z_A} \tag{14.4.7}$$

It is tempting to interpret the power dissipated in the internal impedance of the Thévenin circuit of the receiving antenna (that is, in Z_A) as representing the amount of power re-radiated or scattered by the antenna. However, with the exception of the so-called *minimum-scattering antennas*, such interpretation is not correct.

The issue has been discussed by Silver [21] and more recently in Refs. [634–637]. See also Refs. [610–633] for further discussion of the transmitting, receiving, and scattering properties of antennas.

14.5 Effective Length

The polarization properties of the electric field radiated by an antenna depend on the transverse component of the radiation vector $\mathbf{F}_\perp$ according to Eq. (13.10.5):

$$\mathbf{E} = -jk\eta\frac{e^{-jkr}}{4\pi r}\mathbf{F}_\perp = -jk\eta\frac{e^{-jkr}}{4\pi r}\,(F_\theta\hat{\boldsymbol\theta} + F_\phi\hat{\boldsymbol\phi})$$

The vector *effective length*, or *effective height* of a transmitting antenna is defined in terms of $\mathbf{F}_\perp$ and the input current to the antenna terminals $I_{\rm in}$ as follows [602]:†

$$\mathbf{h} = -\frac{\mathbf{F}_\perp}{I_{\rm in}} \qquad \text{(effective length)} \tag{14.5.1}$$

In general, $\mathbf{h}$ is a function of θ,ϕ. The electric field is, then, written as:

$$\mathbf{E} = jk\eta\frac{e^{-jkr}}{4\pi r}I_{\rm in}\,\mathbf{h} \tag{14.5.2}$$

†Often, it is defined with a positive sign $\mathbf{h} = \mathbf{F}_\perp/I_{\rm in}$.

The definition of $\mathbf{h}$ is motivated by the case of a z-directed Hertzian dipole antenna, which can be shown to have $\mathbf{h} = l\sin\theta\,\hat{\boldsymbol\theta}$. More generally, for a z-directed linear antenna with current $I(z)$, it follows from Eq. (15.1.5) that:

$$\mathbf{h}(\theta) = h(\theta)\,\hat{\boldsymbol\theta}\,, \qquad h(\theta) = \sin\theta\,\frac{1}{I_{\rm in}}\int_{-l/2}^{l/2} I(z')\,e^{jkz'\cos\theta}dz' \tag{14.5.3}$$

As a consequence of the reciprocity principle, it can be shown [602] that the *open-circuit voltage* V at the terminals of a receiving antenna is given in terms of the effective length and the incident field $\mathbf{E}_i$ by:

$$V = \mathbf{E}_i\cdot\mathbf{h} \tag{14.5.4}$$

The normal definition of the effective area of an antenna and the result $G = 4\pi A/\lambda^2$ depend on the assumptions that the receiving antenna is conjugate-matched to its load and that the polarization of the incident wave matches that of the antenna.

The effective length helps to characterize the degree of polarization mismatch that may exist between the incident field and the antenna. To see how the gain-area relationship must be modified, we start with the definition (14.3.1) and use (14.4.5):

$$A(\theta,\phi) = \frac{P_R}{\mathcal{P}_{\rm inc}} = \frac{\frac{1}{2}R_L|I_L|^2}{\frac{1}{2\eta}|\mathbf{E}_i|^2} = \frac{\eta R_L|V|^2}{|Z_L+Z_A|^2|\mathbf{E}_i|^2} = \frac{\eta R_L|\mathbf{E}_i\cdot\mathbf{h}|^2}{|Z_L+Z_A|^2|\mathbf{E}_i|^2}$$

Next, we define the polarization and load *mismatch factors* by:

$$\begin{aligned} e_{\rm pol} &= \frac{|\mathbf{E}_i\cdot\mathbf{h}|^2}{|\mathbf{E}_i|^2\,|\mathbf{h}|^2} \\ e_{\rm load} &= \frac{4R_LR_A}{|Z_L+Z_A|^2} = 1-|\Gamma_{\rm load}|^2\,, \quad \text{where} \quad \Gamma_{\rm load} = \frac{Z_L-Z_A^*}{Z_L+Z_A} \end{aligned} \tag{14.5.5}$$

The effective area can be written then in the form:

$$A(\theta,\phi) = \frac{\eta|\mathbf{h}|^2}{4R_A}\,e_{\rm load}\,e_{\rm pol} \tag{14.5.6}$$

On the other hand, using (14.1.4) and (14.4.1), the power gain may be written as:

$$G(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_T} = \frac{4\pi\dfrac{\eta k^2|\mathbf{F}_\perp|^2}{32\pi^2}}{\dfrac{1}{2}R_A|I_{\rm in}|^2} = \frac{\pi\eta|\mathbf{h}|^2}{\lambda^2 R_A} \quad\Rightarrow\quad \frac{\eta|\mathbf{h}|^2}{4R_A} = \frac{\lambda^2}{4\pi}G(\theta,\phi)$$

Inserting this in Eq. (14.5.6), we obtain the modified area-gain relationship [603]:

$$\boxed{A(\theta,\phi) = e_{\rm load}\,e_{\rm pol}\,\frac{\lambda^2}{4\pi}G(\theta,\phi)} \tag{14.5.7}$$

Assuming that the incident field originates at some antenna with its own effective length $\mathbf{h}_i$, then $\mathbf{E}_i$ will be proportional to $\mathbf{h}_i$ and we may write the polarization mismatch factor in the following form:

$$e_{\rm pol} = \frac{|\mathbf{h}_i\cdot\mathbf{h}|^2}{|\mathbf{h}_i|^2\,|\mathbf{h}|^2} = |\hat{\mathbf{h}}_i\cdot\hat{\mathbf{h}}|^2\,, \qquad \text{where} \quad \hat{\mathbf{h}}_i = \frac{\mathbf{h}_i}{|\mathbf{h}_i|}\,, \quad \hat{\mathbf{h}} = \frac{\mathbf{h}}{|\mathbf{h}|}$$

When the load is conjugate-matched, we have $e_{\text{load}} = 1$, and when the incident field has matching polarization with the antenna, that is, $\hat{\boldsymbol{h}}_i = \hat{\boldsymbol{h}}^*$, then, $e_{\text{pol}} = 1$.

14.6 Communicating Antennas

The communication between a transmitting and a receiving antenna can be analyzed with the help of the concept of gain and effective area. Consider two antennas oriented towards the maximal gain of each other and separated by a distance r, as shown in Fig. 14.6.1.

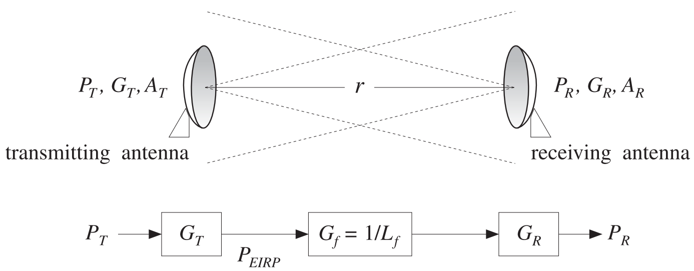

Fig. 14.6.1 Transmitting and receiving antennas.

Let $\{P_T, G_T, A_T\}$ be the power, gain, and effective area of the transmitting antenna, and $\{P_R, G_R, A_R\}$ be the same quantities for the receiving antenna. In the direction of the receiving antenna, the transmitting antenna has $P_{\text{EIRP}} = P_T G_T$ and establishes a power density at distance r:

$$\mathcal{P}_T = \frac{dP_T}{dS} = \frac{P_{\text{EIRP}}}{4\pi r^2} = \frac{P_T G_T}{4\pi r^2} \tag{14.6.1}$$

From the incident power density $\mathcal{P}_T$, the receiving antenna extracts power P_R given in terms of the effective area A_R as follows:

$$\boxed{P_R = A_R \mathcal{P}_T = \frac{P_T G_T A_R}{4\pi r^2}} \qquad \text{(Friis formula)} \tag{14.6.2}$$

This is known as the *Friis formula* for communicating antennas and can be written in several different equivalent forms. Replacing G_T in terms of the transmitting antenna's effective area A_T, that is, $G_T = 4\pi A_T/\lambda^2$, Eq. (14.6.2) becomes:

$$P_R = \frac{P_T A_T A_R}{\lambda^2 r^2} \tag{14.6.3}$$

A better way of rewriting Eq. (14.6.2) is as a product of gain factors. Replacing $A_R = \lambda^2 G_R / 4\pi$, we obtain:

$$P_R = \frac{P_T G_T G_R \lambda^2}{(4\pi r)^2} \tag{14.6.4}$$

The effect of the propagation path, which causes P_R to attenuate with the square of the distance r, can be quantified by defining the *free-space loss* and gain by

$$\boxed{L_f = \left(\frac{4\pi r}{\lambda}\right)^2, \quad G_f = \frac{1}{L_f} = \left(\frac{\lambda}{4\pi r}\right)^2} \qquad \text{(free-space loss and gain)} \tag{14.6.5}$$

Then, Eq. (14.6.4) can be written as the product of the transmit and receive gains and the propagation loss factor:

$$\boxed{P_R = P_T G_T \left(\frac{\lambda}{4\pi r}\right)^2 G_R = P_T G_T \frac{1}{L_f} G_R = P_T G_T G_f G_R} \tag{14.6.6}$$

Such a gain model for communicating antennas is illustrated in Fig. 14.6.1. An additional loss factor, $G_{\text{other}} = 1/L_{\text{other}}$, may be introduced, if necessary, representing other losses, such as atmospheric absorption and scattering. It is customary to express Eq. (14.6.6) additively in dB, where $(P_R)_{\text{dB}} = 10\log_{10} P_R$, $(G_T)_{\text{dB}} = 10\log_{10} G_T$, etc.:

$$(P_R)_{\text{dB}} = (P_T)_{\text{dB}} + (G_T)_{\text{dB}} - (L_f)_{\text{dB}} + (G_R)_{\text{dB}} \tag{14.6.7}$$

Example 14.6.1: A geosynchronous satellite is transmitting a TV signal to an earth-based station at a distance of 40,000 km. Assume that the dish antennas of the satellite and the earth station have diameters of 0.5 m and 5 m, and aperture efficiencies of 60%. If the satellite's transmitter power is 6 W and the downlink frequency 4 GHz, calculate the antenna gains in dB and the amount of received power.

Solution: The wavelength at 4 GHz is $\lambda = 7.5$ cm. The antenna gains are calculated by:

$$G = e_a\left(\frac{\pi d}{\lambda}\right)^2 \quad \Rightarrow \quad G_{\text{sat}} = 263.2 = 24 \text{ dB}, \quad G_{\text{earth}} = 26320 = 44 \text{ dB}$$

Because the ratio of the earth and satellite antenna diameters is 10, the corresponding gains will differ by a ratio of 100, or 20 dB. The satellite's transmitter power is in dB, $P_T = 10\log_{10}(6) = 8$ dBW, and the free-space loss and gain:

$$L_f = \left(\frac{4\pi r}{\lambda}\right)^2 = 4\times 10^{19} \quad \Rightarrow \quad L_f = 196 \text{ dB}, \quad G_f = -196 \text{ dB}$$

It follows that the received power will be in dB:

$$P_R = P_T + G_T - L_f + G_R = 8 + 24 - 196 + 44 = -120 \text{ dBW} \quad \Rightarrow \quad P_R = 10^{-12} \text{ W}$$

or, $P_R = 1$ pW (pico-watt). Thus, the received power is extremely small. □

When the two antennas are mismatched in their polarization with a mismatch factor $e_{\text{pol}} = |\hat{\boldsymbol{h}}_R \cdot \hat{\boldsymbol{h}}_T|^2$, and the receiving antenna is mismatched to its load with $e_{\text{load}} = 1 - |\Gamma_{\text{load}}|^2$, then the Friis formula (14.6.2) is still valid, but replacing A_R using Eq. (14.5.7), leads to a modified form of Eq. (14.6.4):

$$P_R = \frac{P_T G_T G_R \lambda^2}{(4\pi r)^2} |\hat{\boldsymbol{h}}_R \cdot \hat{\boldsymbol{h}}_T|^2 (1 - |\Gamma_{\text{load}}|^2) \tag{14.6.8}$$

14.7 Antenna Noise Temperature

We saw in the above example that the received signal from a geosynchronous satellite is extremely weak, of the order of picowatts, because of the large free-space loss which is typically of the order of 200 dB.

To be able to detect such a weak signal, the receiving system must maintain a noise level that is lower than the received signal. Noise is introduced into the receiving system by several sources.

In addition to the desired signal, the receiving antenna picks up noisy signals from the sky, the ground, the weather, and other natural or man-made noise sources. These noise signals, coming from different directions, are weighted according to the antenna gain and result into a weighted average noise power at the output terminals of the antenna. For example, if the antenna is pointing straight up into the sky, it will still pick up through its sidelobes some reflected signals as well as thermal noise from the ground. Ohmic losses in the antenna itself will be another source of noise.

The antenna output is sent over a feed line (such as a waveguide or transmission line) to the receiver circuits. The lossy feed line will attenuate the signal further and also introduce its own thermal noise.

The output of the feed line is then sent into a low-noise-amplifier (LNA), which pre-amplifies the signal and introduces only a small amount of thermal noise. The low-noise nature of the LNA is a critical property of the receiving system.

The output of the LNA is then passed on to the rest of the receiving system, consisting of downconverters, IF amplifiers, and so on. These subsystems will also introduce their own gain factors and thermal noise.

Such a cascade of receiver components is depicted in Fig. 14.7.1. The sum total of all the noises introduced by these components must be maintained at acceptably low levels (relative to the amplified desired signal.)

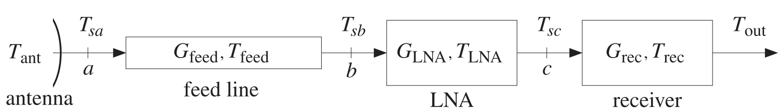

Fig. 14.7.1 Typical receiving antenna system.

The *average power* N (in Watts) of a noise source within a certain bandwidth of B Hz can be quantified by means of an *equivalent temperature* T defined through:

$$\boxed{N = kTB} \quad \text{(noise power within bandwidth } B) \tag{14.7.1}$$

where k is Boltzmann's constant $k = 1.3803\times 10^{-23}$ W/Hz K and T is in degrees Kelvin. The temperature T is not necessarily the physical temperature of the source, it only provides a convenient way to express the noise power. (For a thermal source, T is indeed the physical temperature.) Eq. (14.7.1) is commonly expressed in dB as:

$$N_{dB} = T_{dB} + B_{dB} + k_{dB} \tag{14.7.2}$$

where $T_{dB} = 10\log_{10} T$, $B_{dB} = 10\log_{10} B$, and $k_{dB} = 10\log_{10} k$ is Boltzmann's constant in dB: $k_{dB} = -228.6$ dB. Somewhat incorrectly, but very suggestively, the following units are used in practice for the various terms in Eq. (14.7.2):

$$\text{dB W} = \text{dB K} + \text{dB Hz} + \text{dB W/Hz K}$$

The bandwidth B depends on the application. For example, satellite transmissions of TV signals require a bandwidth of about 30 MHz. Terrestrial microwave links may have B of 60 MHz. Cellular systems may have B of the order of 30 kHz for AMPS or 200 kHz for GSM.

Example 14.7.1: Assuming a 30-MHz bandwidth, we give below some examples of noise powers and temperatures and compute the corresponding signal-to-noise ratio S/N, relative to a 1 pW reference signal ($S = 1$ pW).

T	T_{dB}	$N = kTB$	N_{dB}	S/N
50 K	17.0 dBK	0.0207 pW	−136.8 dBW	16.8 dB
100 K	20.0 dBK	0.0414 pW	−133.8 dBW	13.8 dB
200 K	23.0 dBK	0.0828 pW	−130.8 dBW	10.8 dB
290 K	24.6 dBK	0.1201 pW	−129.2 dBW	9.2 dB
500 K	27.0 dBK	0.2070 pW	−126.8 dBW	6.8 dB
1000 K	30.0 dBK	0.4141 pW	−123.8 dBW	3.8 dB
2400 K	33.8 dBK	1.0000 pW	−120.0 dBW	0.0 dB

The last example shows that 2400 K corresponds to 1 pW noise. □

The average noise power N_{ant} at the antenna terminals is characterized by an equivalent *antenna noise temperature* T_{ant}, such that $N_{ant} = kT_{ant}B$.

The temperature T_{ant} represents the weighted contributions of all the radiating noise sources picked up by the antenna through its mainlobe and sidelobes. The value of T_{ant} depends primarily on the orientation and elevation angle of the antenna, and what the antenna is looking at.

Example 14.7.2: An earth antenna looking at the sky "sees" a noise temperature T_{ant} of the order of 30–60 K. Of that, about 10 K arises from the mainlobe and sidelobes pointing towards the sky and 20–40 K from sidelobes pointing backwards towards the earth [5,650–654]. In rainy weather, T_{ant} might increase by 60 K or more.

The sky noise temperature depends on the elevation angle of the antenna. For example, at an elevations of 5°, 10°, and 30°, the sky temperature is about 20 K, 10 K, and 4 K at 4 GHz, and 25 K, 12 K, and 5 K at 6 GHz [650]. □

Example 14.7.3: The noise temperature of the earth viewed from space, such as from a satellite, is about 254 K. This is obtained by equating the sun's energy that is absorbed by the earth to the thermal radiation from the earth [650]. □

Example 14.7.4: For a base station cellular antenna looking horizontally, atmospheric noise temperature contributes about 70–100 K at the cellular frequency of 1 GHz, and man-made noise contributes another 10–120 K depending on the area (rural or urban). The total value of T_{ant} for cellular systems is in the range of 100–200 K [655,656]. □

In general, a noise source in some direction (θ,ϕ) will be characterized by an effective noise temperature $T(\theta,\phi)$, known as the *brightness temperature*, such that the radiated noise power in that direction will be $N(\theta,\phi)=kT(\theta,\phi)B$. The antenna temperature $T_{\rm ant}$ will be given by the average over all such sources weighted by the receiving gain of the antenna:

$$\boxed{T_{\rm ant}=\frac{1}{\Delta\Omega}\int_0^{\pi}\int_0^{2\pi}T(\theta,\phi)g(\theta,\phi)\,d\Omega} \tag{14.7.3}$$

where $\Delta\Omega$ is the beam solid angle of the antenna. It follows from Eq. (14.2.15) that $\Delta\Omega$ serves as a normalization factor for this average:

$$\Delta\Omega=\int_0^{\pi}\int_0^{2\pi}g(\theta,\phi)\,d\Omega \tag{14.7.4}$$

Eq. (14.7.3) can also be written in the following equivalent forms, in terms of the directive gain or the effective area of the antenna:

$$T_{\rm ant}=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}T(\theta,\phi)D(\theta,\phi)\,d\Omega=\frac{1}{\lambda^2}\int_0^{\pi}\int_0^{2\pi}T(\theta,\phi)A(\theta,\phi)\,d\Omega$$

As an example of Eq. (14.7.3), we consider the case of a *point* source, such as the sun, the moon, a planet, or a radio star. Then, Eq. (14.7.3) gives:

$$T_{\rm ant}=T_{\rm point}\frac{g_{\rm point}\Delta\Omega_{\rm point}}{\Delta\Omega}$$

where $g_{\rm point}$ and $\Delta\Omega_{\rm point}$ are the antenna gain in the direction of the source and the small solid angle subtended by the source. If the antenna's mainlobe is pointing towards that source then, $g_{\rm point}=1$.

As another example, consider the case of a spatially *extended* noise source, such as the sky, which is assumed to have a constant temperature $T_{\rm ext}$ over its angular width. Then, Eq. (14.7.3) becomes:

$$T_{\rm ant}=T_{\rm ext}\frac{\Delta\Omega_{\rm ext}}{\Delta\Omega}\,,\qquad \text{where}\quad \Delta\Omega_{\rm ext}=\int_{\rm ext}g(\theta,\phi)\,d\Omega$$

The quantity $\Delta\Omega_{\rm ext}$ is the portion of the antenna's beam solid angle occupied by the extended source.

As a third example, consider the case of an antenna pointing towards the sky and picking up the atmospheric sky noise through its mainlobe and partly through its sidelobes, and also picking up noise from the ground through the rest of its sidelobes. Assuming the sky and ground noise temperatures are uniform over their spatial extents, Eq. (14.7.3) will give approximately:

$$T_{\rm ant}=T_{\rm sky}\frac{\Delta\Omega_{\rm sky}}{\Delta\Omega}+T_{\rm ground}\frac{\Delta\Omega_{\rm ground}}{\Delta\Omega}$$

where $\Delta\Omega_{\rm sky}$ and $\Delta\Omega_{\rm ground}$ are the portions of the beam solid angle occupied by the sky and ground:

$$\Delta\Omega_{\rm sky}=\int_{\rm sky}g(\theta,\phi)\,d\Omega\,,\qquad \Delta\Omega_{\rm ground}=\int_{\rm ground}g(\theta,\phi)\,d\Omega$$

Assuming that the sky and ground beam solid angles account for the total beam solid angle, we have

$$\Delta\Omega=\Delta\Omega_{\rm sky}+\Delta\Omega_{\rm ground}$$

The sky and ground *beam efficiency* ratios may be defined by:

$$e_{\rm sky}=\frac{\Delta\Omega_{\rm sky}}{\Delta\Omega}\,,\qquad e_{\rm ground}=\frac{\Delta\Omega_{\rm ground}}{\Delta\Omega}\,,\qquad e_{\rm sky}+e_{\rm ground}=1$$

Then, the antenna noise temperature can be written in the form:

$$\boxed{T_{\rm ant}=e_{\rm sky}T_{\rm sky}+e_{\rm ground}T_{\rm ground}} \tag{14.7.5}$$

Example 14.7.5: At 4 GHz and elevation angle of 30°, the sky noise temperature is about 4 K. Assuming a ground temperature of 290 K and that 90% of the beam solid angle of an earth-based antenna is pointing towards the sky and 10% towards the ground, we calculate the effective antenna temperature:

$$T_{\rm ant}=e_{\rm sky}T_{\rm sky}+e_{\rm ground}T_{\rm ground}=0.9\times 4+0.1\times 290=32.6\text{ K}$$

If the beam efficiency towards the sky changes to 85%, then $e_{\rm sky}=0.85$, $e_{\rm ground}=0.15$ and we find $T_{\rm ant}=46.9$ K. □

The mainlobe and sidelobe *beam efficiencies* of an antenna represent the proportions of the beam solid angle occupied by the mainlobe and sidelobe of the antenna. The corresponding beam solid angles are defined by:

$$\Delta\Omega=\int_{\rm tot}g(\theta,\phi)\,d\Omega=\int_{\rm main}g(\theta,\phi)\,d\Omega+\int_{\rm side}g(\theta,\phi)\,d\Omega=\Delta\Omega_{\rm main}+\Delta\Omega_{\rm side}$$

Thus, the beam efficiencies will be:

$$e_{\rm main}=\frac{\Delta\Omega_{\rm main}}{\Delta\Omega}\,,\qquad e_{\rm side}=\frac{\Delta\Omega_{\rm side}}{\Delta\Omega}\,,\qquad e_{\rm main}+e_{\rm side}=1$$

Assuming that the entire mainlobe and a fraction, say α, of the sidelobes point towards the sky, and therefore, a fraction $(1-\alpha)$ of the sidelobes will point towards the ground, we may express the sky and ground beam solid angles as follows:

$$\Delta\Omega_{\rm sky}=\Delta\Omega_{\rm main}+\alpha\Delta\Omega_{\rm side}$$
$$\Delta\Omega_{\rm ground}=(1-\alpha)\Delta\Omega_{\rm side}$$

or, in terms of the efficiency factors:

$$e_{\rm sky}=e_{\rm main}+\alpha e_{\rm side}=e_{\rm main}+\alpha(1-e_{\rm main})$$
$$e_{\rm ground}=(1-\alpha)e_{\rm side}=(1-\alpha)(1-e_{\rm main})$$

Example 14.7.6: Assuming an 80% mainlobe beam efficiency and that half of the sidelobes point towards the sky and the other half towards the ground, we have $e_{\rm main}=0.8$ and $\alpha=0.5$, which lead to the sky beam efficiency $e_{\rm sky}=0.9$. □

14.8 System Noise Temperature

In a receiving antenna system, the signal-to-noise ratio at the receiver must take into account not only the noise picked up by the antenna, and quantified by T_{ant}, but also all the internal noises introduced by the various components of the receiver.

Every device, passive or active, is a source of noise generated internally. Such noise may be modeled as an internal noise source acting at the input of the device, as shown in Fig. 14.8.1. (Alternatively, the noise source can be added at the output, but the input convention is more common.)

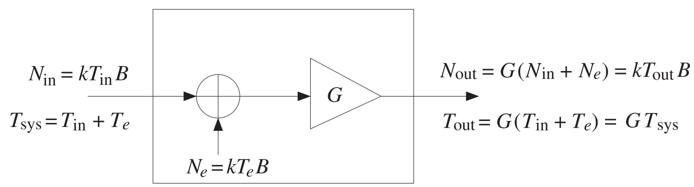

Fig. 14.8.1 Noise model of a device.

The amount of added noise power is expressed in terms of the *effective noise temperature* T_e of the device:

$$\boxed{N_e = kT_eB} \qquad \text{(effective internal noise)} \tag{14.8.1}$$

The sum of N_e and the noise power of the input signal N_{in} will be the total noise power, or the *system noise power* at the input to the device. If the input noise is expressed in terms of its own noise temperature, $N_{in} = kT_{in}B$, we will have:

$$N_{sys} = N_{in} + N_e = k(T_{in} + T_e)B = kT_{sys}B \qquad \text{(total input noise)} \tag{14.8.2}$$

where we introduced the *system noise temperature*† at the device input:

$$\boxed{T_{sys} = T_{in} + T_e} \qquad \text{(system noise temperature)} \tag{14.8.3}$$

If the device has power gain G,‡ then the noise power at the output of the device and its equivalent temperature, $N_{out} = kT_{out}B$, can be expressed as follows:

$$\begin{aligned} N_{out} &= G(N_{in} + N_e) = GN_{sys} \\ T_{out} &= G(T_{in} + T_e) = GT_{sys} \end{aligned} \tag{14.8.4}$$

One interpretation of the system noise power $N_{sys} = kT_{sys}B$ is that it represents the required input power to an equivalent *noiseless* system (with the same gain) that will produce the same output power as the actual noisy system.

If a desired signal with noise power S_{in} is also input to the device, then the signal power at the output will be $S_{out} = GS_{in}$. The *system signal-to-noise ratio* is defined to be the ratio of the input signal power to total system noise power:

† Also called the *operating noise temperature*.

‡ More precisely, G is the *available* power gain of the device, in the notation of Sec. 12.6.

$$\boxed{SNR_{sys} = \frac{S_{in}}{N_{sys}} = \frac{S_{in}}{kT_{sys}B} = \frac{S_{in}}{k(T_{in} + T_e)B}} \qquad \text{(system SNR)} \tag{14.8.5}$$

The SNR is the same whether it is measured at the input or the output of the device; indeed, multiplying numerator and denominator by G and using (14.8.4), we have:

$$SNR_{sys} = \frac{S_{in}}{N_{sys}} = \frac{S_{out}}{N_{out}} \tag{14.8.6}$$

A related concept is that of the *noise figure* of the device, which also characterizes the internally generated noise. It is related to the effective noise temperature T_e by:

$$F = 1 + \frac{T_e}{T_0} \quad \Leftrightarrow \quad T_e = (F - 1)T_0 \tag{14.8.7}$$

where T_0 is the standardized constant temperature $T_0 = 290\,\text{K}$.

The device of Fig. 14.8.1 can be passive or active. The case of a *passive attenuator*, such as a lossy transmission line or waveguide connecting the antenna to the receiver, deserves special treatment.

In this case, the gain G will be less than unity $G < 1$, representing a power loss. For a line of length l and attenuation constant α (nepers per meter), we will have $G = e^{-2\alpha l}$. The corresponding loss factor will be $L = G^{-1} = e^{2\alpha l}$. If $\alpha l \ll 1$, we can write approximately $G = 1 - 2\alpha l$ and $L = 1 + 2\alpha l$.

If the physical temperature of the line is T_{phys} then, from either the input or output end, the line will appear as a thermal noise source of power $kT_{phys}B$. Therefore, the condition $N_{in} = N_{out} = kT_{phys}B$ implies that $kT_{phys}B = Gk(T_{phys} + T_e)B$, which gives:

$$\boxed{T_e = \frac{1}{G}(1 - G)T_{phys} = (L - 1)T_{phys}} \qquad \text{(attenuator)} \tag{14.8.8}$$

If the physical temperature is $T_{phys} = T_0 = 290\,\text{K}$, then, by comparing to Eq. (14.8.7) it follows that the noise figure of the attenuator will be equal to its loss:

$$T_e = (L - 1)T_0 = (F - 1)T_0 \quad \Rightarrow \quad F = L = \frac{1}{G}$$

When the input to the attenuator is an external noise source of power $N_{in} = kT_{in}B$, the system noise temperature at the input and at the output of the attenuator will be:

$$\begin{aligned} T_{sys} &= T_{in} + T_e = T_{in} + (L - 1)T_{phys} \\ T_{out} &= GT_{sys} = GT_{in} + (1 - G)T_{phys} = \frac{1}{L}T_{in} + \left(1 - \frac{1}{L}\right)T_{phys} \end{aligned} \tag{14.8.9}$$

The last equation can be expressed in terms of the input and output powers $N_{in} = kT_{in}B$ and $N_{out} = kT_{out}B$:

$$N_{out} = \frac{1}{L}N_{in} + \left(1 - \frac{1}{L}\right)kT_{phys}B \tag{14.8.10}$$

Thus, the input power is attenuated as expected, but the attenuator also adds its own thermal noise power. More generally, if the input power arises from signal plus noise $P_{\rm in} = S_{\rm in} + N_{\rm in}$, the power at the output will be $P_{\rm out} = S_{\rm out} + N_{\rm out} = GS_{\rm in} + N_{\rm out}$:

$$P_{\rm out} = \frac{1}{L}P_{\rm in} + \left(1 - \frac{1}{L}\right)kT_{\rm phys}B \tag{14.8.11}$$

When two or more devices are cascaded, each will contribute its own internal noise. Fig. 14.8.2 shows two such devices with available power gains G_1 and G_2 and effective noise temperatures T_1 and T_2. The cascade combination can be replaced by an *equivalent* device with gain G_1G_2 and effective noise temperature T_{12}.

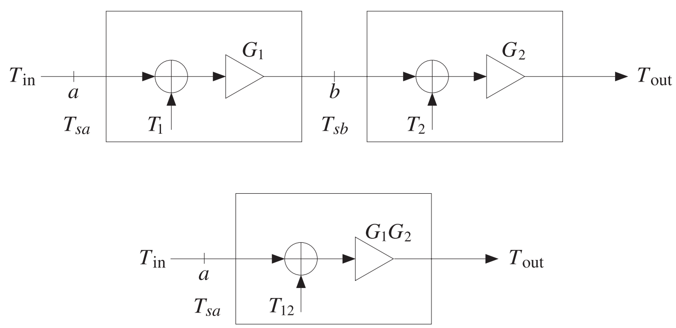

Fig. 14.8.2 Equivalent noise model of two cascaded devices.

The equivalent temperature T_{12} can be determined by superposition. The internal noise power added by the first device, $N_1 = kT_1B$, will go through the gains G_1 and G_2 and will contribute an amount $G_1G_2N_1$ to the output. The noise generated by the second device, $N_2 = kT_2B$, will contribute an amount G_2N_2. The sum of these two powers will be equivalent to the amount contributed to the output by the combined system, $G_1G_2N_{12} = G_1G_2kT_{12}B$. Thus,

$$G_1G_2kT_{12}B = G_1G_2kT_1B + G_2kT_2B \quad \Rightarrow \quad G_1G_2T_{12} = G_1G_2T_1 + G_2T_2$$

It follows that:

$$\boxed{T_{12} = T_1 + \frac{1}{G_1}T_2} \qquad \text{(equivalent noise temperature)} \tag{14.8.12}$$

If G_1 is a large gain, $G_1 \gg 1$, then the contribution of the second device is reduced drastically. On the other hand, if the first device is an attenuator, such as a transmission line, then the contribution of T_2 will be amplified because $G_1 < 1$.

According to Eqs. (14.8.3) and (14.8.4), the system noise temperatures at the overall input, at the output of G_1, and at the overall output will be:

$$\begin{aligned} T_{\rm sys} &= T_{sa} = T_{\rm in} + T_{12} = T_{\rm in} + T_1 + \frac{1}{G_1}T_2 \\ T_{sb} &= G_1T_{sa} = G_1(T_{\rm in} + T_1) + T_2 \\ T_{\rm out} &= G_2T_{sb} = G_1G_2T_{\rm sys} = G_1G_2(T_{\rm in} + T_1) + G_2T_2 \end{aligned} \tag{14.8.13}$$

The system SNR will be:

$$SNR_{\rm sys} = \frac{S_{\rm in}}{kT_{\rm sys}B} = \frac{S_{\rm in}}{k(T_{\rm in} + T_{12})B}$$

The signal powers at points *a*, *b*, and at the output will be $S_a = S_{\rm in}$, $S_b = G_1S_a$, and $S_{\rm out} = G_2S_b = G_1G_2S_a$. It follows from Eq. (14.8.13) that the system SNR will be the *same*, regardless of whether it is referred to the point *a*, the point *b*, or the overall output:

$$SNR_{\rm sys} = SNR_a = SNR_b = SNR_{\rm out}$$

For three cascaded devices, shown in Fig. 14.8.3, any pair of two consecutive ones can be replaced by its equivalent, according to Eq. (14.8.12). For example, the first two can be replaced by T_{12} and then combined with T_3 to give the overall equivalent temperature:

$$T_{12} = T_1 + \frac{1}{G_1}T_2\,, \qquad T_{123} = T_{12} + \frac{1}{G_1G_2}T_3$$

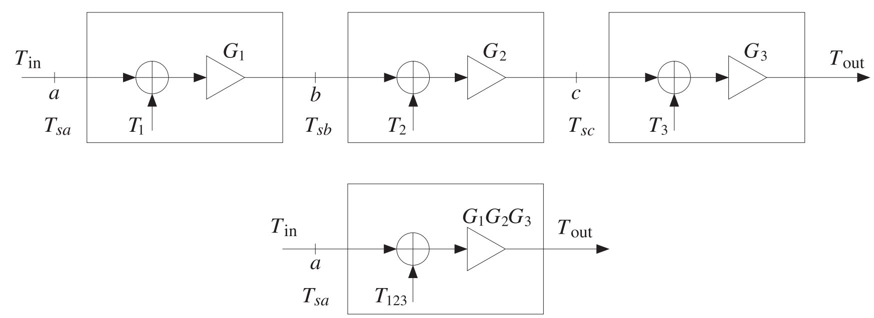

Fig. 14.8.3 Equivalent noise temperatures of three cascaded devices.

Alternatively, we can replace the last two with an equivalent temperature T_{23} and then combine with the first to get:

$$T_{23} = T_2 + \frac{1}{G_2}T_3\,, \qquad T_{123} = T_1 + \frac{1}{G_1}T_{23}$$

From either point of view, we obtain the equivalent temperature:

$$\boxed{T_{123} = T_1 + \frac{1}{G_1}T_2 + \frac{1}{G_1G_2}T_3} \tag{14.8.14}$$

The system SNR will be:

$$SNR_{\rm sys} = \frac{S_{\rm in}}{kT_{\rm sys}B} = \frac{S_{\rm in}}{k(T_{\rm in} + T_{123})B}$$

It is invariant with respect to its reference point:

$$SNR_{\rm sys} = SNR_a = SNR_b = SNR_c = SNR_{\rm out}$$

When expressed in terms of noise figures, Eqs. (14.8.12) and (14.8.14) are also known as Friis's formulas [643], for example, defining the equivalent noise figure as $F_{123} = 1 + T_{123}/T_0$, we have:

$$F_{123} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} \tag{14.8.15}$$

We apply now these results to the antenna receiver shown in Fig. 14.7.1, identifying the three cascaded components as the feed line, the LNA amplifier, and the rest of the receiver circuits. The corresponding noise temperatures are T_{feed}, T_{LNA}, and T_{rec}. The effective noise temperature T_{eff} of the combined system will be:

$$T_{\text{eff}} = T_{\text{feed}} + \frac{1}{G_{\text{feed}}}T_{\text{LNA}} + \frac{1}{G_{\text{feed}}G_{\text{LNA}}}T_{\text{rec}} \tag{14.8.16}$$

Using Eq. (14.8.8), we may replace T_{feed} in terms of the physical temperature:

$$T_{\text{eff}} = \frac{1}{G_{\text{feed}}}(1 - G_{\text{feed}})T_{\text{phys}} + \frac{1}{G_{\text{feed}}}T_{\text{LNA}} + \frac{1}{G_{\text{feed}}G_{\text{LNA}}}T_{\text{rec}} \tag{14.8.17}$$

The input noise temperature T_{in} to this combined system is the antenna temperature T_{ant}. It follows that *system noise temperature*, referred to either the antenna output terminals (point *a*), or to the LNA input (point *b*), will be:

$$T_{\text{sys}} = T_{sa} = T_{\text{ant}} + T_{\text{eff}} = T_{\text{ant}} + \left(\frac{1}{G_{\text{feed}}} - 1\right)T_{\text{phys}} + \frac{1}{G_{\text{feed}}}T_{\text{LNA}} + \frac{1}{G_{\text{feed}}G_{\text{LNA}}}T_{\text{rec}}$$

$$T_{sb} = G_{\text{feed}}T_{sa} = G_{\text{feed}}T_{\text{ant}} + (1 - G_{\text{feed}})T_{\text{phys}} + T_{\text{LNA}} + \frac{1}{G_{\text{LNA}}}T_{\text{rec}}$$

The importance of a high-gain low-noise amplifier is evident from Eq. (14.8.17). The high value of G_{LNA} will minimize the effect of the remaining components of the receiver system, while the small value of T_{LNA} will add only a small amount of noise. Typical values of T_{LNA} can range from 20 K for cooled amplifiers to 100 K at room temperatures.

The feed line can have a major impact. If the line is too lossy or too long, the quantity $G_{\text{feed}} = e^{-2\alpha l}$ will be small, or $1/G_{\text{feed}}$ large, contributing a significant amount to the system noise temperature. Often, the LNA is mounted *before* the feed line, near the focal point of the receiving antenna, so that the effect of the feed line will be suppressed by the factor G_{LNA}.

Similar benefits arise in base station antennas for wireless communications, where high-gain amplifiers can be placed on top of the antenna towers, instead of at the base station, which can be fairly far from the towers [656]. Cable losses in such applications can be in the range 2–4 dB (with gain factors $G_f = 0.63$–0.40.)

The signal to system-noise ratio of the receiving system (referred to point *a* of Fig. 14.7.1) will be the ratio of the received power P_R to the system noise $N_{\text{sys}} = kT_{\text{sys}}B$. Using the Friis formula (for power transmission), we have:

$$SNR = \frac{P_R}{N_{\text{sys}}} = \frac{P_R}{kT_{\text{sys}}B} = (P_T G_T)\frac{1}{L_f}\left(\frac{G_R}{T_{\text{sys}}}\right)\frac{1}{kB} \tag{14.8.18}$$

This ratio is also called the *carrier-to-system-noise ratio* and is denoted by C/N. For a given transmitting EIRP, $P_T G_T$, the receiver performance depends critically on the ratio G_R/T_{sys}, referred to as the G/T ratio of the receiving antenna, or the *figure of merit*. It is usually measured in dB/K. In dB, Eq. (14.8.18) reads as:

$$(SNR)_{\text{dB}} = (P_T G_T)_{\text{dB}} - (L_f)_{\text{dB}} + \left(\frac{G_R}{T_{\text{sys}}}\right)_{\text{dB}} - k_{\text{dB}} - B_{\text{dB}} \tag{14.8.19}$$

The receiver SNR can be also be referred to LNA input (point *b*). The G/T ratio will not change in value, but it will be the ratio of the signal gain after the feed line divided by the system temperature T_{sb}, that is,

$$SNR = (P_T G_T)\frac{1}{L_f}\left(\frac{G_R}{T_{\text{sys}}}\right)\frac{1}{kB} = (P_T G_T)\frac{1}{L_f}\left(\frac{G_R G_{\text{feed}}}{T_{sb}}\right)\frac{1}{kB} \tag{14.8.20}$$

Example 14.8.1: Typical earth-based antennas for satellite communications have G/T ratios of the order of 40 dB/K, whereas satellite receiving antennas can have $G/T = -7$ dB/K or less. The negative sign arises from the smaller satellite antenna gain and the much higher temperature, since the satellite is looking down at a warm earth. □

Example 14.8.2: Consider a receiving antenna system as shown in Fig. 14.7.1, with antenna temperature of 40 K, feed line loss of 0.1 dB, feed line physical temperature of 290 K, LNA gain and effective noise temperature of 50 dB and 80 K. The rest of the receiver circuits have effective noise temperature of 2000 K.

Assuming the receiving antenna has a gain of 45 dB, calculate the system noise temperature and the G/T ratio at point *a* and point *b* of Fig. 14.7.1. Repeat if the feed line loss is 1 dB.

Solution: The feed line has gain $G_{\text{feed}} = 10^{-0.1/10} = 10^{-0.01} = 0.9772$, and the LNA has, $G_{\text{LNA}} = 10^{50/10} = 10^5$. Thus, the system noise temperature at point *a* will be:

$$\begin{aligned} T_{\text{sys}} &= T_{\text{ant}} + \left(\frac{1}{G_{\text{feed}}} - 1\right)T_{\text{phys}} + \frac{1}{G_{\text{feed}}}T_{\text{LNA}} + \frac{1}{G_{\text{feed}}G_{\text{LNA}}}T_{\text{rec}} \\ &= 40 + \left(\frac{1}{10^{-0.01}} - 1\right)290 + \frac{80}{10^{-0.01}} + \frac{2000}{10^{-0.01}\cdot 10^5} \\ &= 40 + 6.77 + 81.87 + 0.02 = 128.62\text{ K} = 21.09\text{ dBK} \end{aligned}$$

At point *b*, we have $T_{sb} = G_{\text{feed}}T_{\text{sys}} = 0.9772 \times 128.62 = 125.69$ K $= 20.99$ dBK. The G/T ratio will be at point *a*, $G_R/T_{\text{sys}} = 45 - 21.09 = 23.91$ dB/K. At point *b* the gain is $G_R G_{\text{feed}} = 45 - 0.1 = 44.9$ dB, and therefore, $G/T = G_R G_{\text{feed}}/T_{sb} = 44.9 - 20.99 = 23.91$ dB/K, which is the same as at point *a*.

For a feed line loss of 1 dB, we find $T_{\text{sys}} = 215.80$ K $= 23.34$ dB. The corresponding G/T ratio will be $45 - 23.34 = 21.66$ dB. □

Example 14.8.3: Suppose the LNA were to be placed in front of the feed line of the above example. Calculate the system noise temperature in this case when the feed line loss is 0.1 dB and 1 dB.

Solution: Interchanging the roles of the feed line and the LNA in Eq. (14.8.16), we have for the system noise temperature:

$$T_{\rm sys} = T_{\rm ant} + T_{\rm LNA} + \frac{1}{G_{\rm LNA}}T_{\rm feed} + \frac{1}{G_{\rm feed}G_{\rm LNA}}T_{\rm rec}$$

With $G_{\rm feed} = 10^{-0.1/10} = 0.9772$, we find $T_{\rm feed} = 6.75$ K, and with $G_{\rm feed} = 10^{-1/10} = 0.7943$, $T_{\rm feed} = 75.1$ K. Because of the large LNA gain, the value of $T_{\rm sys}$ will be essentially equal to $T_{\rm ant} + T_{\rm LNA}$; indeed, we find in the two cases:

$$T_{\rm sys} = 120.0205 \text{ K} \quad \text{and} \quad T_{\rm sys} = 120.0259 \text{ K}$$

The G/T will be $45 - 10\log_{10}(120) = 20.8$ dB/K. □

14.9 Data Rate Limits

The system SNR limits the data rate between the two antennas. According to Shannon's theorem, the maximum data rate (in bits/sec) that can be achieved is:

$$C = B\log_2(1 + SNR) \qquad \text{(Shannon's channel capacity)} \tag{14.9.1}$$

where *SNR* is in absolute units. For data rates $R \le C$, Shannon's theorem states that there is an ideal coding scheme that would guarantee error-free transmission.

In a practical digital communication system, the bit-error probability or *bit-error rate* (BER), P_e, is small but not zero. The key performance parameter from which P_e can be calculated is the ratio E_b/N_0, where E_b is the energy per bit and N_0 is the system noise spectral density $N_0 = kT_{\rm sys}$.

The functional relationship between P_e and E_b/N_0 depends on the particular digital modulation scheme used. For example, in binary and quadrature phase-shift keying (BPSK and QPSK), P_e and its inverse are given by [653]:

$$P_e = \frac{1}{2}{\rm erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \quad \Leftrightarrow \quad \frac{E_b}{N_0} = \big[{\rm erfinv}(1-2P_e)\big]^2 \tag{14.9.2}$$

where ${\rm erfc}(x)$ is the complementary error function, and ${\rm erf}(x)$ and ${\rm erfinv}(x)$ are the error function and its inverse as defined in MATLAB:

$${\rm erfc}(x) = 1 - {\rm erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\,, \qquad y = {\rm erf}(x) \;\Leftrightarrow\; x = {\rm erfinv}(y) \tag{14.9.3}$$

The relationships (14.9.2) are plotted in Fig. 14.9.1. The left graph also shows the ideal Shannon limit $E_b/N_0 = \ln 2 = 0.6931 \equiv -1.5917$ dB, which is obtained by taking the limit of Eq. (14.9.1) for infinite bandwidth.

If T_b is the time it takes to transmit one bit, then the data rate will be $R = 1/T_b$, and the required power, $P = E_b/T_b = E_bR$. It follows that the SNR will be:

$$SNR = \frac{P}{N_{\rm sys}} = \frac{P}{kT_{\rm sys}B} = \frac{E_b}{N_0}\,\frac{R}{B}$$

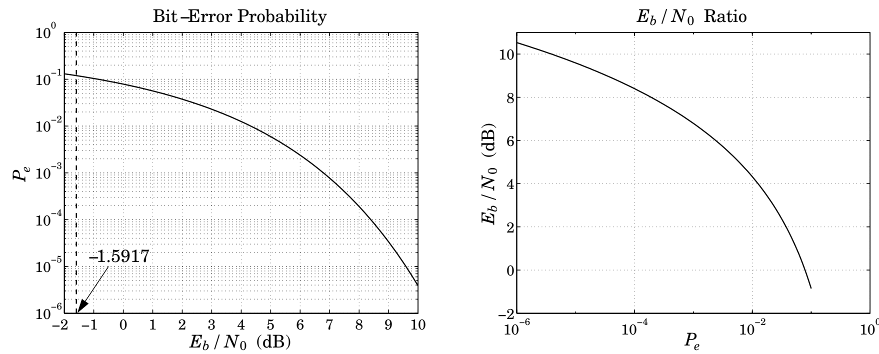

Fig. 14.9.1 P_e versus E_b/N_0, and its inverse, for a BPSK system.

Using the small-x expansion, $\log_2(1+x) \simeq x/\ln 2$, Shannon's condition for error-free transmission becomes in the limit $B \to \infty$:

$$R \le C = B\log_2\left(1 + \frac{E_b}{N_0}\frac{R}{B}\right) \to B\,\frac{E_bR}{N_0B\ln 2} = \frac{R}{\ln 2}\frac{E_b}{N_0} \quad \Rightarrow \quad \frac{E_b}{N_0} \ge \ln 2 = -1.5917 \text{ dB}$$

For a pair of communicating antennas, the received power will be related to the energy per bit by $P_R = E_b/T_b = E_bR$. Using Friis's formula, we find:

$$R\,\frac{E_b}{N_0} = \frac{P_R}{N_0} = \frac{P_{\rm EIRP}\,G_f\,G_R}{kT_{\rm sys}} = (P_TG_T)\,\frac{G_R}{kT_{\rm sys}}\left(\frac{\lambda}{4\pi r}\right)^2 \tag{14.9.4}$$

which may be solved for the maximum achievable data rate (in bits/sec):

$$R = \frac{1}{E_b/N_0}\,\frac{P_{\rm EIRP}\,G_f\,G_R}{kT_{\rm sys}} = \frac{1}{E_b/N_0}\,(P_TG_T)\,\frac{G_R}{kT_{\rm sys}}\left(\frac{\lambda}{4\pi r}\right)^2 \tag{14.9.5}$$

An overall gain factor, $G_{\rm other} = 1/L_{\rm other}$, may be introduced representing other losses, such as atmospheric losses.

Example 14.9.1: The Voyager spacecrafts have antenna diameter and aperture efficiency of $d = 3.66$ m (12 ft) and $e_a = 0.6$. The operating frequency is $f = 8.415$ GHz and the transmitter power $P_T = 18$ W. Assuming the same efficiency for the 70-m receiving antenna at NASA's deep-space network at Goldstone, CA, we calculate the antenna gains using the formula $G = e_a(\pi d/\lambda)^2$, with $\lambda = c/f = 0.0357$ m:

$$G_T = 47.95 \text{ dB}, \quad G_R = 73.58 \text{ dB}, \quad P_T = 13.62 \text{ dBW}$$

Assuming a system noise temperature of $T_{\rm sys} = 25$ K $= 13.98$ dBK for the receiving antenna, we find for the noise spectral density $N_0 = kT_s = -214.62$ dBW/Hz, where we used $k = -228.6$ dB. Assuming a bit-error rate of $P_e = 5{\times}10^{-3}$, we find from Eq. (14.9.2) the required ratio $E_b/N_0 = 3.317 = 5.208$ dB.

Voyager 1 was at Jupiter in 1977, at Saturn in 1980, and at Neptune in 1989. In 2002 it was at a distance of about $r = 12{\times}10^9$ km. It is expected to be at $r = 22{\times}10^9$ km in the year

2020. We calculate the corresponding free-space gain $G_f = (\lambda/4\pi r)^2$ and the expected data rates R from Eq. (14.9.5), where r is in units of 10^9 km:

location	r	G_f (dB)	R (dB)	R (bits/sec)
Jupiter	0.78	−288.78	50.78	119,757
Saturn	1.43	−294.05	45.52	35,630
Neptune	4.50	−304.01	35.56	3,598
2002	12.00	−312.53	27.04	506
2020	22.00	−317.79	21.78	150

where we assumed an overall loss factor of $G_{\text{other}} = -5$ dB. More information on the Voyager mission and NASA's deep-space network antennas can be obtained from the web sites [852] and [853]. □

14.10 Satellite Links

Consider an earth-satellite-earth system, as shown in Fig. 14.10.1. We wish to establish the total link budget and signal to system-noise ratio between the two earth antennas communicating via the satellite.

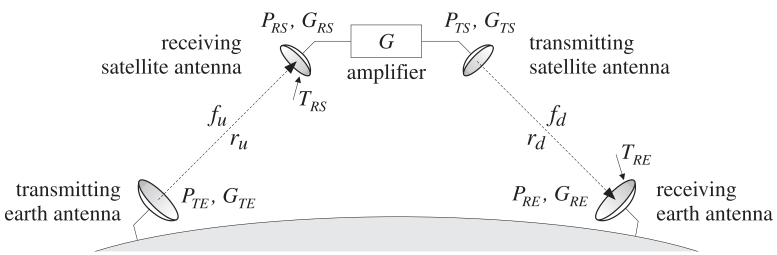

Fig. 14.10.1 Uplink and downlink in satellite communications.

In a geosynchronous satellite system, the uplink/downlink frequencies f_u, f_d are typically 6/4 GHz or 14/11 GHz. The distances r_u, r_d are of the order of 40000 km. Let $\lambda_u = c/f_u$ and $\lambda_d = c/f_d$ be the uplink/downlink wavelengths. The free-space gain/loss factors will be from Eq. (14.6.5):

$$G_{fu} = \frac{1}{L_{fu}} = \left(\frac{\lambda_u}{4\pi r_u}\right)^2, \quad G_{fd} = \frac{1}{L_{fd}} = \left(\frac{\lambda_d}{4\pi r_d}\right)^2 \tag{14.10.1}$$

The satellite has an on-board amplifier with gain G, which could be as high as 100–120 dB. Using Friis formula in its gain form, Eq. (14.6.6), the link equations for the uplink, the satellite amplification, and the downlink stages can be written as follows:

$$\begin{aligned}
P_{EIRP} &= P_{TE}\,G_{TE} && \text{(EIRP of transmitting earth antenna)}\\
P_{RS} &= P_{TE}\,G_{TE}\,G_{fu}\,G_{RS} && \text{(received power by satellite antenna)}\\
P_{TS} &= G P_{RS} && \text{(transmitted power by satellite antenna)}\\
P_{RE} &= P_{TS}\,G_{TS}\,G_{fd}\,G_{RE} && \text{(received power by earth antenna)}
\end{aligned}$$

Expressing P_{RE} in terms of P_{TE}, we have:

$$P_{RE} = P_{RS}\,G\,G_{TS}\,G_{fd}\,G_{RE} = P_{TE}\,G_{TE}\,G_{fu}\,G_{RS}\,G\,G_{TS}\,G_{fd}\,G_{RE} \tag{14.10.2}$$

or, showing the free-space loss factors explicitly:

$$P_{RE} = P_{TE}\,G_{TE}\,G_{RS}\,G\,GTS\,G_{RE}\left(\frac{\lambda_u}{4\pi r_u}\right)^2\left(\frac{\lambda_d}{4\pi r_d}\right)^2 \tag{14.10.3}$$

Because there are two receiving antennas, there will be two different system noise temperatures, say T_{RS} and T_{RE}, for the satellite and earth receiving antennas. They incorporate the antenna noise temperatures as well as the receiver components. The corresponding figures of merit will be the quantities G_{RS}/T_{RS} and G_{RE}/T_{RE}. We may define the uplink and downlink SNR's as the signal-to-system-noise ratios for the individual antennas:

$$SNR_u = \frac{P_{RS}}{kT_{RS}B}, \quad SNR_d = \frac{P_{RE}}{kT_{RE}B} \tag{14.10.4}$$

The system noise T_{RS} generated by the receiving satellite antenna will get amplified by G and then transmitted down to the earth antenna, where it will contribute to the system noise temperature. By the time it reaches the earth antenna it will have picked up the gain factors $G\,G_{TS}\,G_{fd}\,G_{RE}$. Thus, the net system noise temperature measured at the receiving earth antenna will be:

$$T_{\text{sys}} = T_{RE} + G\,G_{TS}\,G_{fd}\,G_{RE}T_{RS} \tag{14.10.5}$$

The SNR of the total link will be therefore,

$$SNR_{\text{tot}} = \frac{P_{RE}}{kT_{\text{sys}}B} \tag{14.10.6}$$

$$\begin{aligned}
SNR_{\text{tot}}^{-1} &= \frac{k(T_{RE} + G\,G_{TS}\,G_{fd}\,G_{RE}T_{RS})B}{P_{RE}} = \frac{kT_{RE}B}{P_{RE}} + \frac{k\,G\,G_{TS}\,G_{fd}\,G_{RE}\,T_{RS}B}{P_{RE}}\\
&= \frac{kT_{RE}B}{P_{RE}} + \frac{k\,G\,G_{TS}\,G_{fd}\,G_{RE}T_{RS}B}{G\,G_{TS}\,G_{fd}\,G_{RE}P_{RS}} = \frac{kT_{RE}B}{P_{RE}} + \frac{kT_{RS}B}{P_{RS}} = SNR_d^{-1} + SNR_u^{-1}
\end{aligned}$$

where we used Eq. (14.10.2). It follows that:

$$\boxed{SNR_{\text{tot}} = \frac{1}{SNR_u^{-1} + SNR_d^{-1}}} \tag{14.10.7}$$

This is also written in the form:

$$\left(\frac{C}{N}\right)_{\text{tot}} = \frac{1}{\left(\frac{C}{N}\right)_u^{-1} + \left(\frac{C}{N}\right)_d^{-1}}$$

Example 14.10.1: As an example of a link budget calculation, assume the following data: The uplink/downlink distances are 36000 km. The uplink/downlink frequencies are 6/4 GHz. The diameters of the earth and satellite antennas are 15 m and 0.5 m with 60% aperture efficiencies. The earth antenna transmits power of 1 kW and the satellite transponder gain is 90 dB. The satellite receiving antenna is looking down at an earth temperature of 300 K and has a noisy receiver of effective noise temperature of 2700 K, whereas the earth receiving antenna is looking up at a sky temperature of 50 K and uses a high-gain LNA amplifier of 80 K (feedline losses may be ignored.) The bandwidth is 30 MHz.

The uplink and downlink wavelengths are $\lambda_u = 0.05$ m and $\lambda_d = 0.075$ m, corresponding to 6 and 4 GHz. The up and down free-space gains and losses are:

$$G_{fu} = -L_{fu} = -199.13 \text{ dB}, \quad G_{fd} = -L_{fd} = -195.61 \text{ dB}$$

The antenna gains are calculated to be:

$$G_{TE} = 57.27 \text{ dB}, \quad G_{RS} = 27.72 \text{ dB}, \quad G_{TS} = 24.20 \text{ dB}, \quad G_{RE} = 53.75 \text{ dB}$$

With $P_{TE} = 1$ kW $= 30$ dBW, the EIRP of the transmitting earth antenna will be: $P_{\text{EIRP}} = 30 + 57.27 = 87.27$ dBW. The power received by the satellite antenna will be:

$$P_{RS} = 87.27 - 199.13 + 27.72 = -84.14 \text{ dBW}$$

After boosting this up by the transponder gain of 90 dB, the power transmitted down to the receiving earth antenna will be:

$$P_{TS} = 90 - 84.14 = 5.86 \text{ dBW}$$

The EIRP of the transmitting satellite antenna will be $(P_{TS}G_{TS})_{\text{dB}} = 5.86 + 24.20 = 30.06$ dBW. The downlink power received by the earth antenna will be:

$$P_{RE} = 30.06 - 195.61 + 53.75 = -111.80 \text{ dBW}$$

The system noise temperatures are: $T_{RS} = 300 + 2700 = 3000$ K and $T_{RE} = 50 + 80 = 130$ K, and in dBK: $T_{RS} = 34.77$ and $T_{RE} = 21.14$. The 30 MHz bandwidth is in dB: $B_{\text{dB}} = 10\log_{10}(30\times 10^6) = 74.77$ dB Hz. Using the Boltzmann constant k in dB, $k_{\text{dB}} = -228.6$, we calculate the receiver system noise powers in dB, using $N = k_{\text{dB}} + T_{\text{dB}} + B_{\text{dB}}$:

$$N_{RS} = -228.6 + 34.77 + 74.77 = -119.06 \text{ dBW}$$
$$N_{RS} = -228.6 + 21.14 + 74.77 = -132.69 \text{ dBW}$$

It follows that the G/T ratios and system SNR's for the receiving antennas will be:

$$(G/T)_u = G_{RS} - T_{RS} = 27.72 - 34.77 = -7.05 \text{ dB/K}$$

$$(G/T)_d = G_{RE} - T_{RE} = 53.75 - 21.14 = 32.61 \text{ dB/K}$$

$$SNR_u = P_{RS} - N_{RS} = -84.14 + 119.06 = 34.92 \text{ dB} = 3103.44$$

$$SNR_d = P_{RE} - N_{RE} = -111.80 + 132.69 = 20.89 \text{ dB} = 122.72$$

The overall system SNR is calculated from Eq. (14.10.7) using absolute units:

$$SNR_{\text{tot}} = \frac{1}{SNR_u^{-1} + SNR_d^{-1}} = \frac{1}{(3103.44)^{-1} + (122.72)^{-1}} = 118.05 = 20.72 \text{ dB}$$

The overall SNR is essentially equal to the downlink SNR. □

14.11 Radar Equation

Another example of the application of the concepts of gain and effective area and the use of Friis formulas is radar. Fig. 14.11.1 shows a radar antenna, which illuminates a target at distance r in the direction of its maximal gain. The incident wave on the target will be reflected and a portion of it will be intercepted back at the antenna.

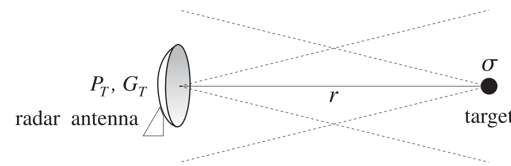

Fig. 14.11.1 Radar antenna and target.

The concept of *radar cross section* σ provides a measure of the effective area of the target and the re-radiated power. If the radar antenna transmits power P_T with gain G_T, the power density of the transmitted field at the location of the target will be:

$$\boxed{\mathcal{P}_T = \frac{P_T G_T}{4\pi r^2}} \tag{14.11.1}$$

From the definition of σ, the power intercepted by the target and re-radiated is:

$$\boxed{P_{\text{target}} = \sigma \mathcal{P}_T = \frac{P_T G_T \sigma}{4\pi r^2}} \tag{14.11.2}$$

By definition of the radar cross section, the power P_{target} will be re-radiated isotropically and establish a power density back at the location of the radar antenna:

$$\boxed{\mathcal{P}_{\text{target}} = \frac{P_{\text{target}}}{4\pi r^2} = \frac{P_T G_T \sigma}{(4\pi r^2)^2}} \tag{14.11.3}$$

The amount of power received by the radar antenna will be given in terms of its effective area A_R as follows:

$$\boxed{P_R = A_R \mathcal{P}_{\text{target}} = \frac{P_T G_T A_R \sigma}{(4\pi)^2 r^4}} \quad \text{(radar equation)} \tag{14.11.4}$$

This is also known as Friis' formula. Using $A_R = A_T$ and $G_T = 4\pi A_T/\lambda^2$, we may express Eq. (14.11.4) in the alternative forms:

$$P_R = \frac{P_T A_T^2 \sigma}{4\pi \lambda^2 r^4} = \frac{P_T G_T^2 \lambda^2 \sigma}{(4\pi)^3 r^4} = P_T G_T^2 \left(\frac{\lambda}{4\pi r}\right)^4 \left(\frac{4\pi\sigma}{\lambda^2}\right) \tag{14.11.5}$$

Introducing the equivalent target gain corresponding to the radar cross section, that is, $G_\sigma = 4\pi\sigma/\lambda^2$, we may also write Eq. (14.11.5) as the product of gains:

$$P_R = P_T G_T^2 G_f^2 G_\sigma \tag{14.11.6}$$

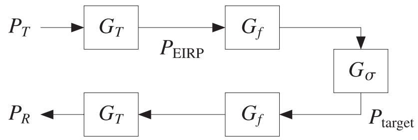

Fig. 14.11.2 Gain model of radar equation.

Fig. 14.11.2 shows this gain model. There are two free-space paths and two antenna gains, acting as transmit and receive gains.

The minimum detectable received power, $P_{R,\rm min}$, defines the maximum distance $r_{\rm max}$ at which the target can be detected:

$$P_{R,\rm min} = \frac{P_T G_T A_R \sigma}{(4\pi)^2 r_{\rm max}^4}$$

Solving for $r_{\rm max}$, we obtain:

$$\boxed{r_{\rm max} = \left[\frac{P_T G_T A_T \sigma}{(4\pi)^2 P_{R,\rm min}}\right]^{1/4}} \qquad \text{(radar range)} \qquad (14.11.7)$$

If the target is not in the direction of maximal gain G_T of the antenna, but in some other direction, say (θ,ϕ), then the maximal gain G_T in Eq. (14.11.5) must be replaced with $G_T g(\theta,\phi)$, where $g(\theta,\phi)$ is the antenna's normalized gain. The received power can be expressed then as:

$$P_R = \frac{P_T G_T^2 g^2(\theta,\phi)\lambda^2\sigma}{(4\pi)^3 r^4} \qquad (14.11.8)$$

In ground-based air search radars trying to detect approaching aircraft flying at a fixed height h, the power received by the radar can be made to be *independent* of the distance r, within a certain distance range, by choosing the gain $g(\theta,\phi)$ appropriately. As shown in Fig. 14.11.3, the height h is related to r by $h = r\cos\theta$.

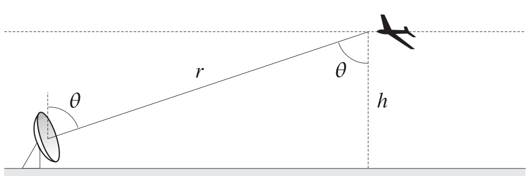

Fig. 14.11.3 Secant antenna gain.

If the gain is designed to have the *secant-squared* shape $g(\theta,\phi) = K/\cos^2\theta$, where K is a constant, then the power will become independent of r. Indeed,

$$P_R = \frac{P_T G_T^2 g^2(\theta,\phi)\lambda^2\sigma}{(4\pi)^3 r^4} = \frac{P_T G_T^2 K^2\lambda^2\sigma}{(4\pi)^3 r^4\cos^4\theta} = \frac{P_T G_T^2 K^2\lambda^2\sigma}{(4\pi)^3 h^4}$$

The secant behavior is not valid over all polar angles θ, but only over a certain range, such as $0 \le \theta \le \theta_{\rm max}$, where $\theta_{\rm max}$ corresponds to the maximum range of the radar $r_{\rm max} = h/\cos\theta_{\rm max}$. The desired secant shape can be achieved by appropriate feeds of the radar dish antenna, or by an antenna array with properly designed array factor. In Sec. 19.5, we present such a design for an array.

14.12 Problems