

MEAN AND RMS CALCULATIONS FOR SAMPLED PERIODIC SIGNALS WITH NON-INTEGER NUMBER OF SAMPLES PER PERIOD APPLIED TO AC ENERGY SYSTEMS

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***Abstract.** Measuring and monitoring energy parameters of AC power systems requires several calculations, such RMS values of voltage and current, plus active and reactive power. These calculations are performed with samples of the monitored waveforms, at sample frequency equal to N times the nominal frequency of waves. Conventional algorithms regard N as constant and integer. When the actual frequency is not exactly the nominal, errors are introduced in the calculations, and averages are used over several cycles to reduce these errors. However, in protection systems, these calculations must be performed every cycle for quick response of protection circuits and averages are impracticable. The purpose of this article is to introduce a method to minimize these errors making N variable and non-integer.*

***Keywords:** Periodic sinusoidal signals, RMS value, AC power, AC power sampling, Active AC power.*

1. INTRODUCTION

In AC power energy systems, measurements of transferred energy must be performed at each network point. These measurements are expressed by waveform parameters of voltage and current, such the root-mean-square (RMS) values, for characterizing of each voltage or current waveforms, and crossed products, to calculate the transferred power (Albu et al., 2003).

Due the characteristics of AC energy, voltage and current waveforms mean are zero; however, during abnormal operation fluctuations may occur or oftentimes the offset error introduced by transducers must be accounted for the correct characterization of the energy.

These calculations are performed with a digital processing, using sequential samples with sample frequency equal to N times the AC frequency, nominally 60 Hz or 50 Hz. If the value N exactly matches the ratio between sample frequency $f_s = 1/T_s$ and AC frequency $f = 1/T$, there are only quantization errors due to finite length of bit number of samples. If, however, the ratio f_s/f is not an integer value, what occurs most of the cases, more errors are introduced (Mesrobian et al., 1991). Usually the averaging process along several cycles significantly reduces those errors. However in protection systems these calculations need to be performed every cycle, or even in half cycle to allow fast decision for a device to open or switch a certain circuit. This study was conducted to eliminate, or reduce the errors in calculations with samples from one cycle (or half cycle).

It is not discussed here the methods to determine the samples of voltages and currents in phase or quadrature, nor which crossed products result the active and reactive power, but only the methods to perform the calculations. Also it is not discussed the measurement of period T ; this information or the relation f_s/f must be known a priori to perform the calculations of the proposed method.

2. MATHEMATICAL MODELS

The values that need to be calculated are by definition the mean and the mean square over one period of a continuous (and periodic) signal $x(t)$.

$$m(x) = \frac{1}{T} \int_T x(t) dt \quad (1)$$

$$m(x^2) = \frac{1}{T} \int_T x^2(t) dt \quad (2)$$

The discrete method usually employed for calculating them is presented below, as described by the current standard (IEEE, 1995).

$$\bar{X} = \frac{1}{N} \sum_{n=0}^{N-1} x_n \quad (3)$$

$$\overline{X^2} = \frac{1}{N} \sum_{n=0}^{N-1} x_n^2 \quad (4)$$

Where, the values x_n are the samples of the periodic signal and exact N samples are equally spaced along one complete signal period.

These expressions work well only when the period T of the signal is equal to N times the sample period T_s . In this case, the start and stop phases of the sampled period are the same.

The mean of crossed product between two sampled periodic signals with the same period is given analogously.

$$\overline{XY} = \frac{1}{N} \sum_{n=0}^{N-1} (x_n y_n) \quad (5)$$

Where, the values of x_n and y_n are the samples of the two periodic signals.

2.1 Frequency displacement

When the supposed integer value N does not match the ratio between the period T of the periodic signal and the sample period T_s a greater or smaller one exact cycle will be summed leading on an inaccurate estimate. This fact occurs when the frequency of the periodic signal is not exactly the nominal value. The samples do not repeat from one cycle to another, and summation over one cycle cannot be made with exact N samples.

Calculations of mean, root-mean-square and crossed products made with N samples will result in values above or below the true value. In the long term, the average of these values converges to the true values, but in short term, there are errors. This fact makes impracticable the calculation of these values cycle by cycle for fast decision with sufficient reliability.

The mean and root-mean-square calculated values oscillate around the true values with a period corresponding to the "beating" of the frequencies f of the signal and f_s/N supposed by the sampling. The amplitude of this oscillation varies with the difference between two frequencies. Since f_s is fixed by design and f could fluctuate around a nominal value or even change severely during abnormal conditions, for precise calculations, N should be allowed to be variable and non-integer.

3. CALCULATIONS WITH NON-INTEGER AMOUNT OF SAMPLES

When the period T of sampled periodic signal is not an integer multiple of sample period T_s , there is not an integer number N matching the ratio between them. So, N should be allowed to be non-integer.

Another advantage of N being non-integer is that the sample frequency f_s doesn't need to be an integer multiple of nominal frequency f of the signal, allowing use of any value of f_s for system design.

A non-integer value of N will be interpreted as having the last sample summed with non-unitary weight, because it corresponds to an interval with a fraction of the sample period T_s . Also, this fractional sample doesn't correspond to any actual sample, and need to be calculated by interpolation between the preceding and the trailing samples. This interpolation will be taken linearly for speed. The interpolation error should result small due to the great number of samples per period, and the objective of characterizing global parameter, concerning a complete cycle.

3.1 Mean by summation with linear interpolation

For a periodic signal represented by a sequence of N samples, corresponding to one complete cycle, the calculation of the mean by summation with linear interpolation of the values between each two samples is given by the following equation.

$$\bar{X} = \frac{1}{N} \sum_{n=0}^{N-1} \frac{x_n + x_{n+1}}{2} \quad (6)$$

Where, N is the integer number of samples per period, and the values x_n are the samples. Equation (6) results a weight $1/2$ for the first and last samples, and unity weight for the others, using $N+1$ samples.

$$\bar{X} = \frac{1}{N} \left(\frac{x_0}{2} + \sum_{n=1}^{N-1} x_n + \frac{x_N}{2} \right) \quad (7)$$

This calculation demands a differentiated calculation for the first and last samples. To facilitate the calculation, they are defined the interpolation samples below.

$$x_{n+1/2} = \frac{x_n + x_{n+1}}{2} \quad (8)$$

So, the mean by summation with linear interpolation becomes the following.

$$\bar{X} = \frac{1}{N} \sum_{n=0}^{N-1} x_{n+1/2} \quad (9)$$

3.2 Interpolation for a fractional sample period

When the period T of the periodic signal is not an integer multiple of sample period T_s , there is not an integer number matching the ratio between T and T_s , and may be defined the period P .

$$P = \frac{T}{T_s} = N + \mathbf{d} \quad (10)$$

Here, N is the integer part of P , and $0 \leq \mathbf{d} < 1$ is the fractional part. In this case, there are N whole sample periods, plus a fraction \mathbf{d} of last sample period, demanding interpolation. For each whole sample period, from 0 to $N-1$, the value of interpolated sample $x_{n+1/2}$ is calculated, corresponding to the instants $(n+1/2)T_s$, or samples $n+1/2$, using Eq. (8). For the last, fractional sample period, the interpolated sample is given by the following.

$$x_{N+\mathbf{d}/2} = \left(1 - \frac{\mathbf{d}}{2} \right) x_N + \frac{\mathbf{d}}{2} x_{N+1} \quad (11)$$

Equation (11) calculates the linear interpolation between the points (N, x_N) and $(N+1, x_{N+1})$, at $N+\mathbf{d}/2$, which corresponds to the center of interval from N to $N+\mathbf{d}$. Thus, the mean from

summation with linear interpolation for non-integer numbers of samples is given by the equation below.

$$\bar{X} = \frac{1}{P} \left(\sum_{n=0}^{N-1} x_{n+1/2} + \mathbf{d} x_{N+\mathbf{d}/2} \right) \quad (12)$$

The sum represents the contribution of whole sample periods, with unitary weight, and the last term represents the fractional sample period with weight \mathbf{d} . The total weight is P , due to Eq. (10).

3.3 Sample adjustments

The calculation presented in Eq. (12) demands $N+1$ interpolations. If, however, the samples are displaced half sample period T_s , these samples, x_n , becomes exactly the interpolated values, for the intervals $(n-1/2)T_s$ to $(N+1/2)T_s$. Thus, each sample must be displaced by a time $T_s/2$, resulting the following equations.

$$\bar{X} = \frac{1}{P} \left(\sum_{n=1}^N x_n + \mathbf{d} x_{N+(\mathbf{d}+1)/2} \right) \quad (13)$$

$$x_{N+(\mathbf{d}+1)/2} = \frac{1-\mathbf{d}}{2} x_N + \frac{1+\mathbf{d}}{2} x_{N+1} \quad (14)$$

This calculation demands only one interpolation, at the last, fractional sample period. The sum is made with the existing samples x_n , quickly done by a simple processor.

Equation (14) shows the needed interpolation to calculate the fractional sample period of the end of period. The final expression is the following.

$$\bar{X} = \frac{1}{P} \left(\sum_{n=1}^N x_n + \mathbf{d} \left(\frac{1-\mathbf{d}}{2} x_N + \frac{1+\mathbf{d}}{2} x_{N+1} \right) \right) \quad (15)$$

3.4 Calculation of the root-mean-square (RMS) value

To evaluate the root-mean-square value, the procedure is analogous. The last sample is obtained by interpolation. The only difference is to substitute the samples x_n by its squares x_n^2 in Eq. (13), maintaining linear interpolation to determine the fractional sample $x_{N+(\mathbf{d}+1)/2}$. The following equations give the calculation for RMS value of sampled periodic signals with non-integer number of samples per period.

$$\overline{X^2} = \frac{1}{P} \left(\sum_{n=1}^N x_n^2 + \mathbf{d} (x_{N+(\mathbf{d}+1)/2})^2 \right) \quad (16)$$

$$X_{RMS} = \sqrt{\frac{1}{P} \left(\sum_{n=1}^N x_n^2 + \mathbf{d} \left(\frac{1-\mathbf{d}}{2} x_N + \frac{1+\mathbf{d}}{2} x_{N+1} \right)^2 \right)} \quad (17)$$

3.5 Crossed products

Crossed product between two sampled periodic signals x_n and y_n with the same period are made in a similar way as the RMS evaluation. It is necessary to take into account the interpolations for the second signal y_n . The fractional sample is given by changes in Eq. (14).

$$y_{N+(d+1)/2} = \frac{1-d}{2} y_N + \frac{1+d}{2} y_{N+1} \quad (18)$$

The mean of crossed product is given by modifying Eq. (16) into the following.

$$\overline{XY} = \frac{1}{P} \left(\sum_{n=1}^N (x_n y_n) + d x_{N+(d+1)/2} y_{N+(d+1)/2} \right) \quad (19)$$

3.6 Expressions to be used for calculations

All calculations needed are characterization of two signals individually and crossed, with mean, root-mean-square and mean of crossed product. These calculations are performed by the expressions listed below.

The first step is to determine the values of N and d , integer and fractional part of P , ratio between the period T and the sample period T_s , using Eq. (10).

The second step is to calculate the interpolated samples of the last, fractional sample period, for each signal, using Eq. (14) and Eq. (18).

The third step is to calculate the sums, using Eq. (13), Eq. (16), Eq. (19) and the following.

$$\overline{Y} = \frac{1}{P} \left(\sum_{n=1}^N y_n + d y_{N+(d+1)/2} \right) \quad (20)$$

$$\overline{Y^2} = \frac{1}{P} \left(\sum_{n=1}^N y_n^2 + d (y_{N+(d+1)/2})^2 \right) \quad (21)$$

The fourth and last step is to obtain the final values, from the sums, using the equations below.

$$X_{DC} = \overline{X} \quad (22)$$

$$Y_{DC} = \overline{Y} \quad (23)$$

$$X_{RMS} = \sqrt{\overline{X^2}} \quad (24)$$

$$Y_{RMS} = \sqrt{\overline{Y^2}} \quad (25)$$

To obtain the RMS values excluding the DC (mean) values, the equations are presented below. In order to distinguish these RMS values from the above, these are called X_{AC} and Y_{AC} , because the DC values are excluded.

$$X_{AC} = \sqrt{\overline{X^2} - \bar{X}^2} \quad (26)$$

$$Y_{AC} = \sqrt{\overline{Y^2} - \bar{Y}^2} \quad (27)$$

$$XY_{AC} = \overline{XY} - \bar{X} \bar{Y} \quad (28)$$

The value XY_{AC} may be the active power, for example.

4. RESULTS

Several implementations of the presented expressions were simulated with 64, 100 and 128 samples per nominal period, for up to 50% of frequency displacement, applied to sinusoidal signals of several initial phases and up to 10% of offset (DC value). A sinusoidal signal with third and fifth harmonics, DC offset and random noise was used to simulate the application in an actual case. The example below shows the results with a 20% third harmonic, 4% fifth harmonic, 10% offset and 1% random gaussian noise over the main sinusoidal signal, with 128 samples per nominal cycle. The total AC amplitude was adjusted to unitary true AC RMS value. This signal is a good example of the highly distorted signals acquired in AC power systems, at output of the acquisition circuitry.

The frequency of main signal was changed from 50% to 150% of the nominal value, in 1% steps. The initial phase was changed from 0 (zero) to 360 degrees of the main signal. Figure 1 shows an example of signal, with normalized frequency of 1.25, and initial phase 315 degrees.

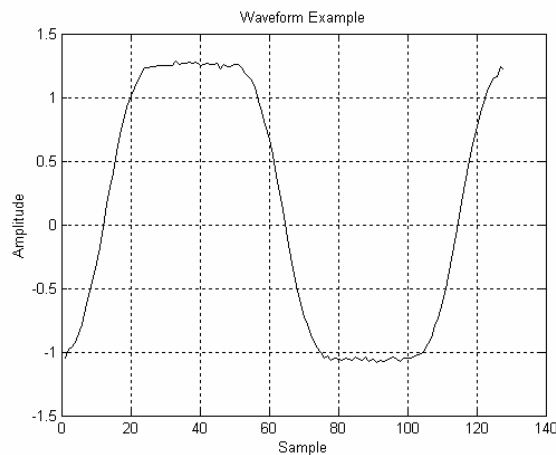


Figure 1 - A sample of the signal used for simulations

For each frequency, DC, AC RMS and total RMS values were calculated over all initial phases by conventional method with 128 samples and the proposed method. Maximum and minimum values for each frequency were plotted. Figure 2 shows the results for calculations

of DC value with 128 samples, according to Eq. (3), regardless the actual frequency of signal, and according to Eq. (15), regarding the actual frequency of signal. Figure 3 repeats the results according for the proposed method, in an amplified scale. Note that the errors were less than 2% of total signal amplitude while for conventional method the errors were more than 60% of total signal amplitude.

For the nominal frequency, both methods provided similar results, because the equivalence between initial and final phases. There are compensations for the fixed samples method, and its result is very close to the true value, regardless the initial phase.

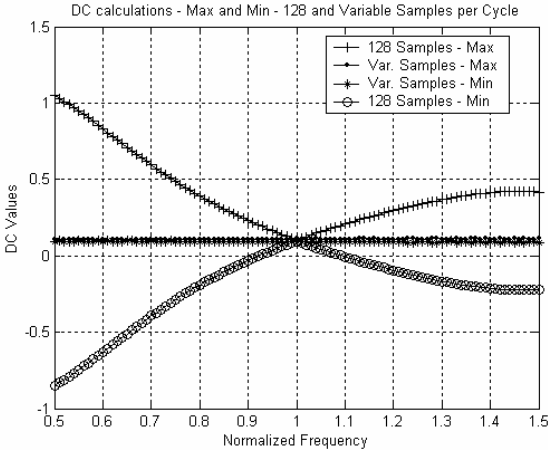


Figure 2 - DC value with 128 and variable samples

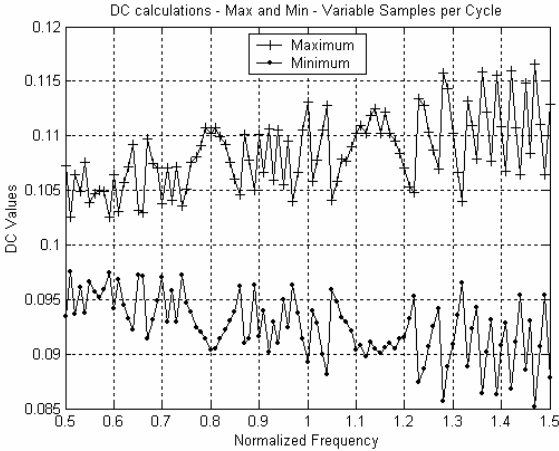


Figure 3 - DC value with variable samples

Figure 4 shows the results of AC RMS calculations according to Eq. (26), over fixed 128 samples and variable samples. Figure 5 repeats the results for variable samples, in an amplified scale.

The errors were more than 50% of the total signal amplitude for fixed 128 samples and less than 1% for variable samples. As in Fig. 2, for the nominal frequency, both methods provided similar results.

Figure 6 and Fig. 7 show the same set of results for calculations of the total RMS value. Figure 6 compares conventional method based on Eq. (4) and the proposed method based on Eq. (17). Figure 7 displays the results in an amplified scale. The proposed method introduces significantly less errors than conventional method with fixed 128 samples.

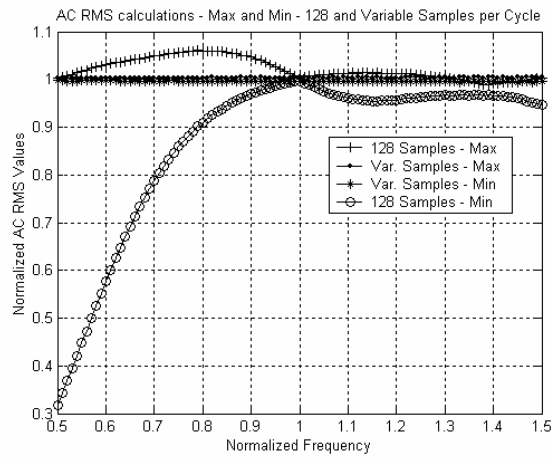


Figure 4 - AC RMS value with 128 and variable samples

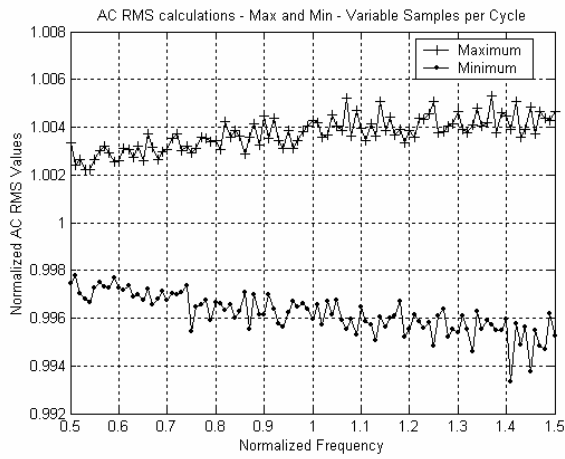


Figure 5 - AC RMS value with variable samples

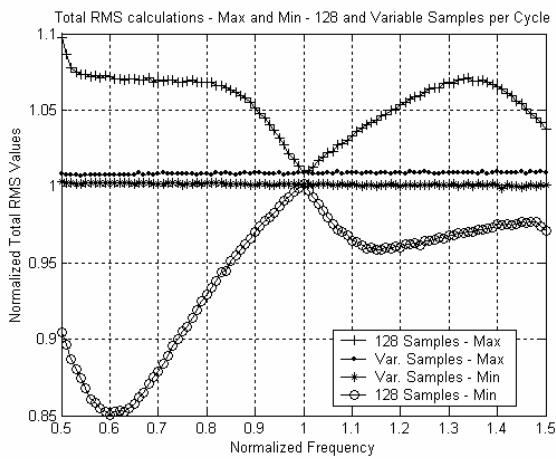


Figure 6 - Total RMS value with 128 and variable samples

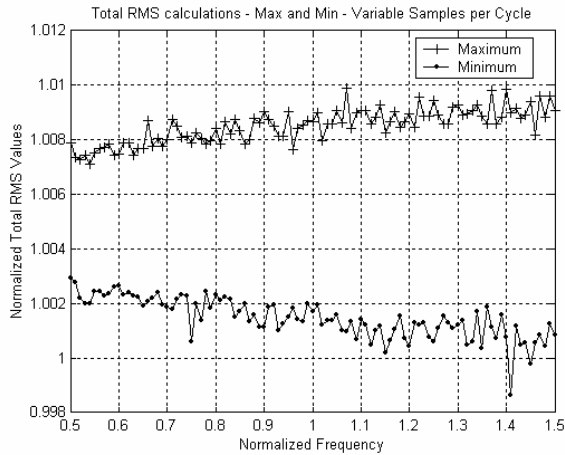


Figure 7 - Total RMS value with variable samples

The following tables summarize the maximum errors obtained in several simulations, with the save wave of Fig. 1, for various maximum frequency displacements, above and below the nominal frequency, and for 128, 100 and 64 samples per nominal cycle. Table 1 shows the results for 128 samples, Table 2 for 100 samples, and Table 3, for 64 samples per nominal cycle. Note the small error in the target calculation (AC RMS) is less than the total noise introduced (1%) for simulations. For the DC calculation the errors were greater, because the main value is less (10%).

Table 1: Maximum errors for 128 samples per nominal cycle

Samples		128	non-integer	128	non-integer
$\Delta f/f$	Normalized Freq.	DC error		AC RMS error	
1%	0,99 1,01	17%	11%	0,89%	0,45%
2%	0,98 1,02	32%	11%	1,3%	0,47%
5%	0,95 1,05	73%	11%	2,8%	0,51%
10%	0,90 1,10	150%	13%	5,9%	0,51%
20%	0,80 1,20	320%	13%	15%	0,51%
50%	0,50 1,50	920%	15%	59%	0,64%

Table 2: Maximum errors for 100 samples per nominal cycle

Samples		100	non-integer	100	non-integer
$\Delta f/f$	Normalized Freq.	DC error		AC RMS error	
1%	0,99 1,01	18%	19%	0,89%	0,59%
2%	0,98 1,02	31%	19%	1,4%	0,60%
5%	0,95 1,05	76%	19%	2,9%	0,64%
10%	0,90 1,10	150%	19%	6,0%	0,64%
20%	0,80 1,20	320%	19%	15%	0,64%
50%	0,50 1,50	920%	19%	60%	0,69%

Table 3: Maximum errors for 64 samples per nominal cycle

$\Delta f/f$	Samples		64	non-integer	64	non-integer
	Normalized Freq.		DC error		AC RMS error	
1%	0,99	1,01	20%	18%	0,99%	0,69%
2%	0,98	1,02	32%	20%	1,5%	0,71%
5%	0,95	1,05	74%	20%	3,1%	0,72%
10%	0,90	1,10	150%	22%	6,0%	0,79%
20%	0,80	1,20	320%	23%	15%	0,85%
50%	0,50	1,50	920%	23%	60%	0,85%

5. CONCLUSION

Conventional methods for calculating DC and RMS values with fixed number of samples may introduce very large errors when signal frequency deviates from nominal value. By using the presented method with variable and non integer number of samples, it is possible to do almost errorless calculation of mean and root-mean-square values of sampled periodic signals by using non-integer number of samples per period. The error introduced by this method is always very small, about 1%, comparable to quantization errors due to A/D converters.

The calculations to apply this method are easily implemented in any simple Digital Signal Processor, and may be executed in real time letting a decision device to know the values in a per cycle basis.

This method is also helpful for system design allowing a choice of the sampling frequency regardless the exact value of the signal frequency thus making them independent from one to another.

Sample rates (64, 100 and 128 samples per cycle) showed weak influence on AC RMS errors, for both methods. Higher sample rates are better; however, the number of 64 samples per cycle with the proposed method provided satisfactory results.

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