Introduction to bond graph theory

First part: basic concepts
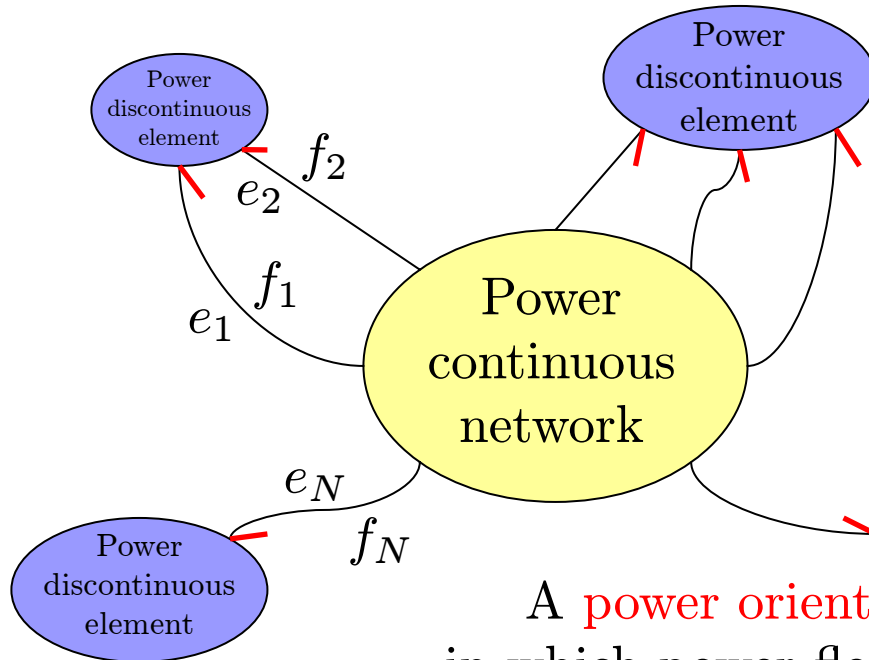
References

■ D.C. Karnopp, D.L. Margolis & R.C. Rosenberg, *System Dynamics. Modeling and Simulation of Mechatronic Systems* (3rd edition). Wiley (2000). ISBN: 0-471-33301-8.

■ B.M. Maschke, A.J. van der Schaft & P.C. Breedveld, An intrinsic Hamiltonian formulation of the dynamics of *LC*-circuits. *IEEE Trans. Circ. & Systems I* **42**, pp. 73-82 (1995).

■ G. Golo, P.C. Breedveld, B.M. Maschke & A.J. van der Schaft, Input output representations of Dirac structures and junction structures in bond graphs. *Proc. of the 14th Int. Symp. of Mathematical Theory of Networks and Systems* (MTNS2000), Perpignan, June 19-23 (2000):

http://www.univ-perp.fr/mtns2000/articles/B01.pdf

Network description of systems

Power discontinuous element

Power discontinuous element

Power discontinuous element

Power continuous network

f_2

e_2

f_1

e_1

e_N

f_N

$$[e_i][f_i] = \text{power}, \ i = 1, \ldots, N$$

$$\text{flows} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \in V$$

$$\text{efforts} \quad e = (e_1, \ldots, e_N) \in V^*$$

A power orientation stroke sets the way in which power flows when $e_i f_i > 0$. We adopt an input power convention, except when indicated.

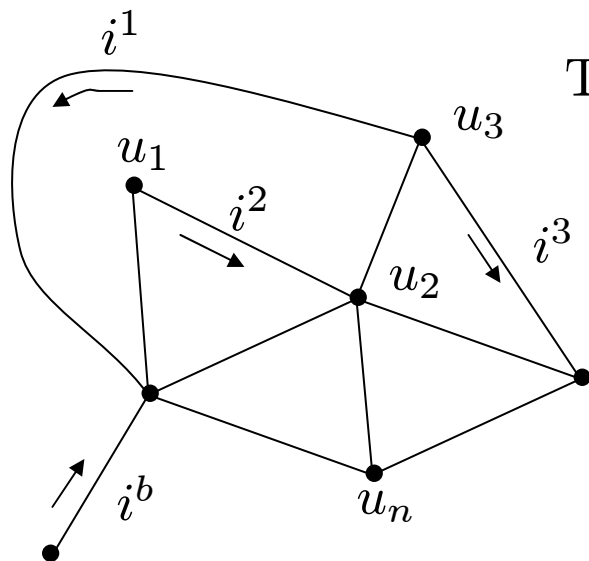$$e(f) \equiv \langle e, f \rangle = \sum_{i=1}^{N} e_i f_i \in K \quad (\mathbb{R} \text{ or } \mathbb{C})$$

The network is power continuous if it establishes relations such that

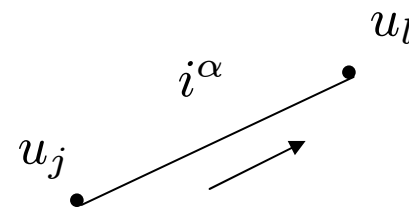$$\langle e, f \rangle = 0$$

Example: Tellegen's theorem

Circuit with b branches and n nodes

To each node we assign a voltage u_j, $j = 1, \ldots, n$

To each branch we assign
a current i^α, $\alpha = 1, \ldots, b$,
and this gives an orientation to the branch

For each branch we define
the voltage drop v_α, $\alpha = 1, \ldots, b$:

This is KVL!

$$v_\alpha = u_j - u_l$$

Mathematically, the circuit, with the orientation induced by the currents, is a digraph (directed graph)

We can define its $n \times b$ adjacency matrix A by

$$A_\alpha^i = \begin{cases} -1 & \text{if branch } \alpha \text{ is incident on node } i \\ +1 & \text{if branch } \alpha \text{ is anti-incident on node } i \\ 0 & \text{otherwise} \end{cases}$$

Then, KCL states that

$$\sum_{\alpha=1}^{b} A_\alpha^i i^\alpha = 0, \qquad \forall \ i = 1, \ldots, n$$

In fact, KVL can also be stated in terms of A:

$$v_\alpha = \sum_{i=1}^{n} A_\alpha^i u_i$$

The sum contains only two terms, because each branch connects only two nodes

Tellegen's theorem. Let $\{v_{(1)\alpha}(t_1)\}_{\alpha=1,\ldots,b}$ be a set of branch voltages satisfying KVL at time t_1, and let $\{i^\alpha_{(2)}(t_2)\}_{\alpha=1,\ldots,b}$ be a set of currents satisfying KCL at time t_2. Then

$$\sum_{\alpha=1}^{b} v_{(1)\alpha}(t_1) i^\alpha_{(2)}(t_2) \equiv \langle v_{(1)}(t_1), i_{(2)}(t_2) \rangle = 0$$

Proof:

KVL

$$\sum_{\alpha=1}^{b} v_{(1)\alpha}(t_1) i^\alpha_{(2)}(t_2) = \sum_{\alpha=1}^{b} \left(\sum_{i=1}^{n} A^i_\alpha u_{(1)i}(t_1) \right) i^\alpha_{(2)}(t_2)$$

$$= \sum_{i=1}^{n} \left(\sum_{\alpha=1}^{b} A^i_\alpha i^\alpha_{(2)}(t_2) \right) u_{(1)i}(t_1) = \sum_{i=1}^{n} 0 \cdot u_{(1)i}(t_1) = 0$$

KCL

Notice that $\{v_{(1)\alpha}(t_1)\}$ and $\{i^{\alpha}_{(2)}(t_2)\}$ may correspond to different times and they may even correspond to different elements for the branches of the circuit.

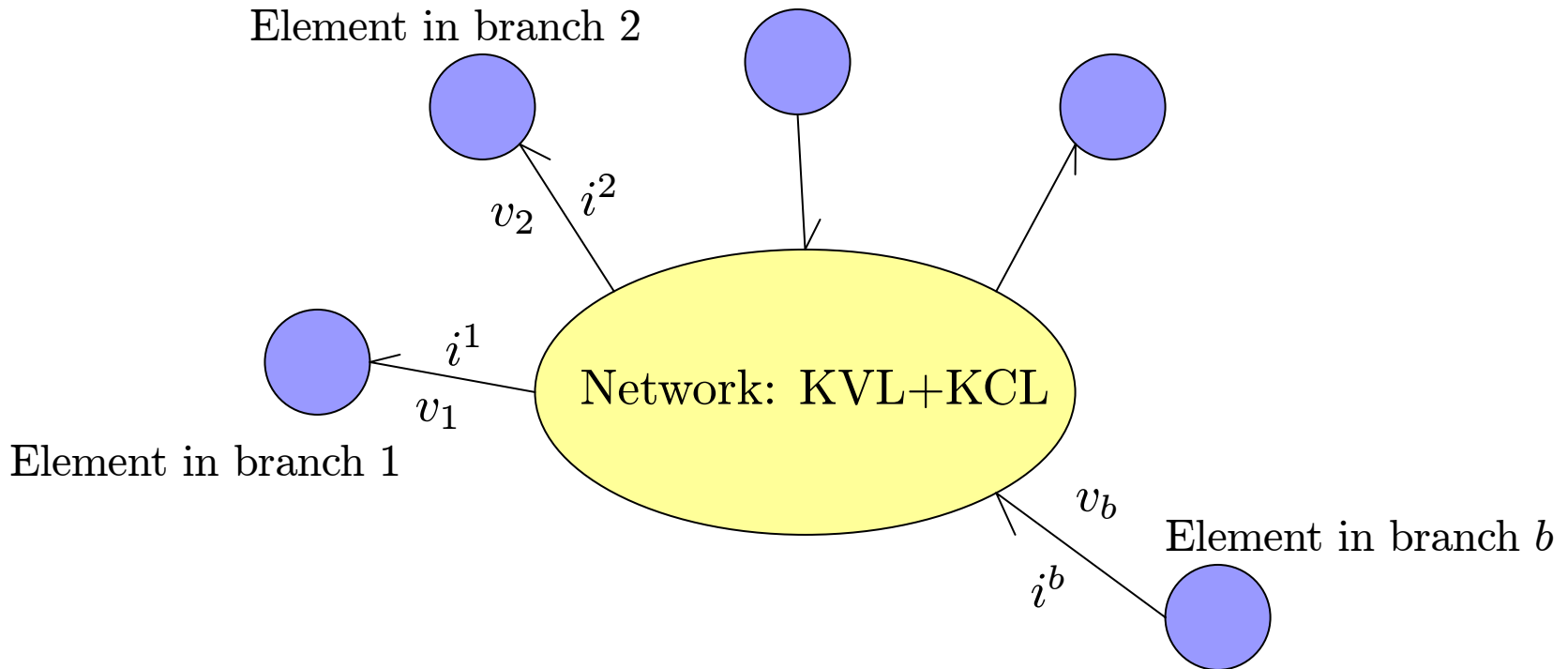The only invariant element is the topology of the circuit *i.e.* the adjacency matrix.

Corollary. Under the same conditions as for Tellegen's theorem,

$$\left\langle \frac{\mathrm{d}^r}{\mathrm{d}t_1^r} v_{(1)}(t_1), \frac{\mathrm{d}^s}{\mathrm{d}t_2^s} i_{(2)}(t_2) \right\rangle = 0$$

for any $r, s \in \mathbb{N}$.

In fact, even duality products between voltages and currents in different domains (time or frequency) can be taken and the result is still zero.

In terms of abstract network theory, a circuit can be represented as follows

Element in branch 2

v_2 i^2

i^1

Network: KVL+KCL

v_1

Element in branch 1

v_b

Element in branch b

i^b

The kth branch element imposes a
constitutive relation
between v_k and i^k.

May be linear or nonlinear,
algebraic or differential, . . .

Basic bond graph elements

In bond graph theory, every element,
power continuous or not, is represented by a multiport.

Ports are connected by bonds.

The basic blocs of standard bond graph theory are

1-ports:

C-type elements	Integral relation between f and e	
I-type elements	Integral relation between e and f	
R-type elements	Algebraic relation between f and e	power discontinuous
Effort sources	Fixes e independently of f	
Flow sources	Fixes f independently of e	

2-ports: Transformers
          Gyrators

3-ports: 0-junctions
          1-junctions

power continuous, make up the network
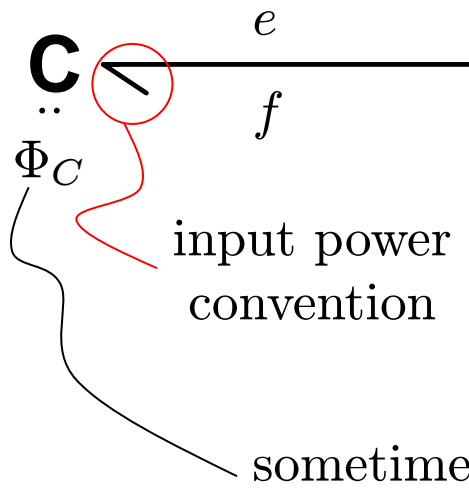
C-type elements

Constitutive relation through a state variable q called displacement.

C ← $\underset{f}{\overset{e}{\longleftarrow}}$

Φ_C

input power convention

sometimes indicated this way
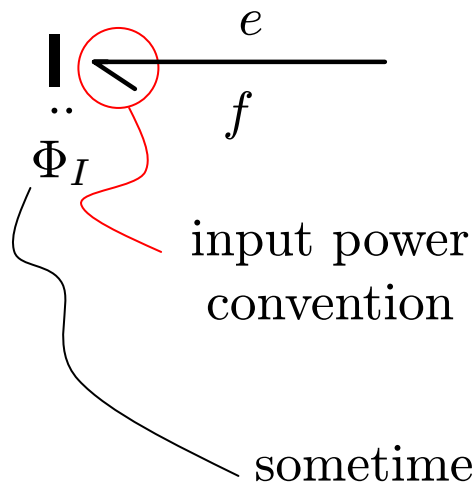
$$\dot{q} = f$$

$$e = \Phi_C^{-1}(q)$$

C-type elements have a preferred computational direction, from f to e:

$$e(t) = (e(t_0) - \Phi_C^{-1}(0)) + \Phi_C^{-1}\left(\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right)$$

Examples: mechanical springs and electric capacitors

Linear case: $\qquad \Phi_C^{-1}(q) = \dfrac{q}{C}$

I-type elements

Constitutive relation through a state variable p called momentum.

$$\dot{p} = e$$

$$f = \Phi_I^{-1}(p)$$

sometimes indicated this way

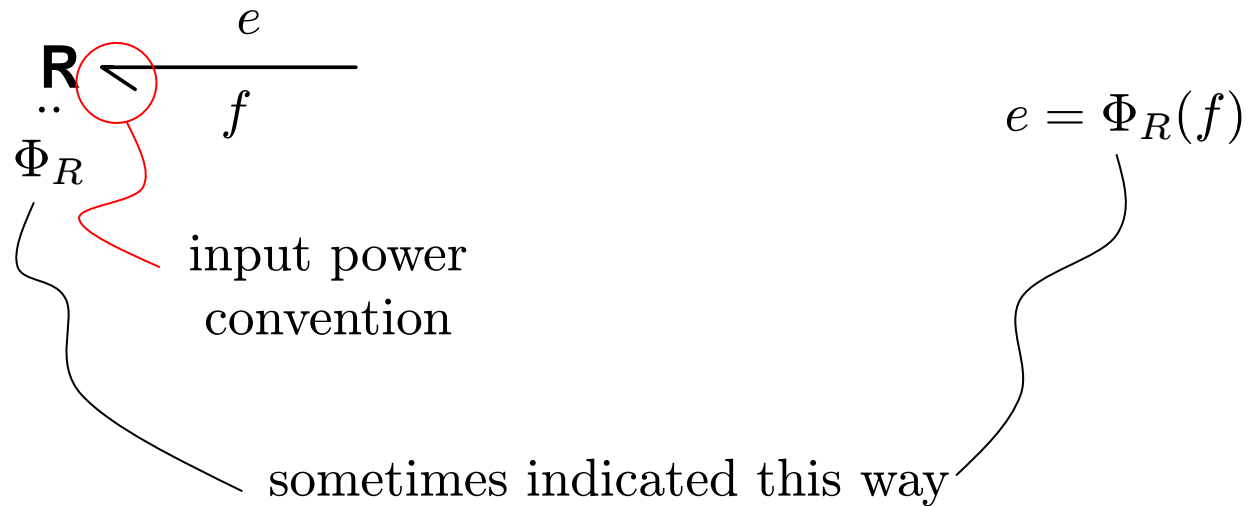I-type elements have a preferred computational direction, from e to f:

$$f(t) = (f(t_0) - \Phi_I^{-1}(0)) + \Phi_I^{-1}\left(\int_{t_0}^{t} e(\tau)\ \mathrm{d}\tau\right)$$

Examples: mechanical masses and electric inductors

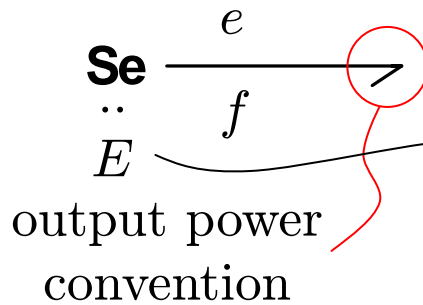Linear case: $\qquad \Phi_I^{-1}(p) = \dfrac{p}{I}$

R-type elements

Direct algebraic constitutive relation between e and f.

$$e = \Phi_R(f)$$

input power convention

sometimes indicated this way

Examples: electric resistor, viscous mechanical damping, static torque-velocity relationships

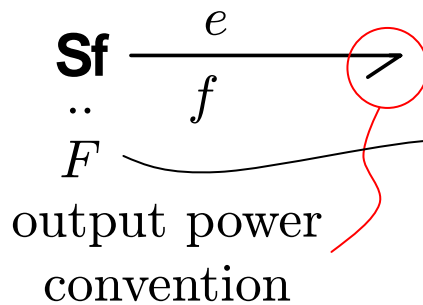Linear case: $\qquad\qquad \Phi_R(f) = Rf$

Effort sources

e does not depend on f

Se $\xrightarrow[\;f\;]{\;e\;}$

$\ddot{E}$

$e = E(t)$

output power convention

f is given by the system to which the source is connected

Flow sources

f does not depend on e

Sf $\xrightarrow[\;f\;]{\;e\;}$

$\ddot{F}$

$f = F(t)$

output power convention

e is given by the system to which the source is connected

Transformers

$$e_1 = \tau \cdot e_2$$
$$\tau \cdot f_1 = f_2$$

output power convention

input power convention

TF

$\dfrac{e_1}{f_1}$ $\dfrac{e_2}{f_2}$

τ

transformer modulus $\tau > 0$
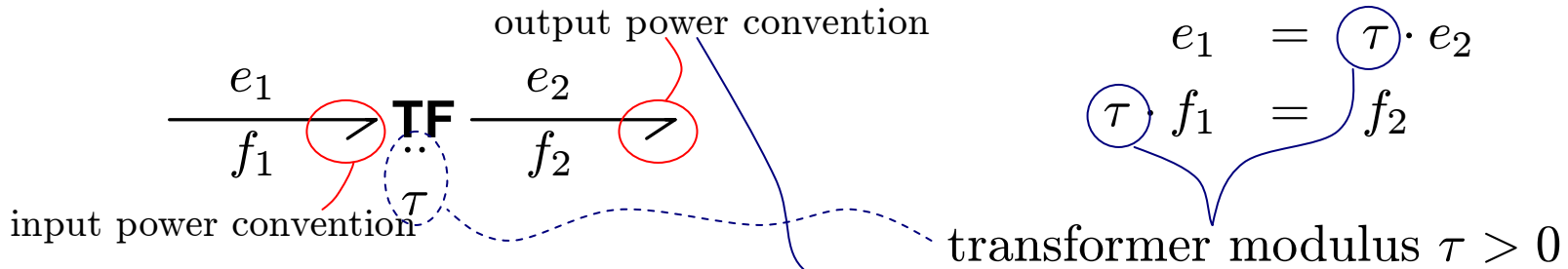
It is power continuous: $e_1 f_1 - e_2 f_2 = 0$

Gyrators

$$e_1 = \tau \cdot f_2$$
$$\tau \cdot f_1 = e_2$$

output power convention

input power convention

GY

$\dfrac{e_1}{f_1}$ $\dfrac{e_2}{f_2}$

τ

gyrator modulus $\tau > 0$
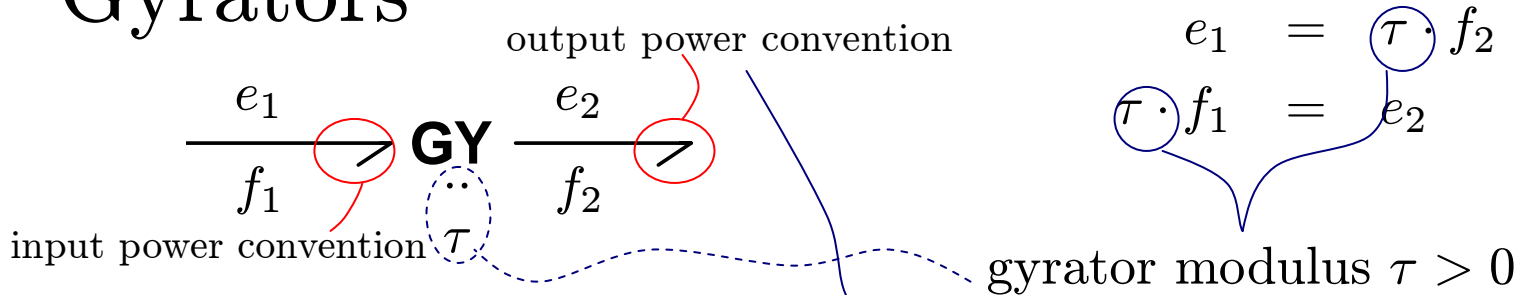
It is power continuous: $e_1 f_1 - e_2 f_2 = 0$

0-junctions
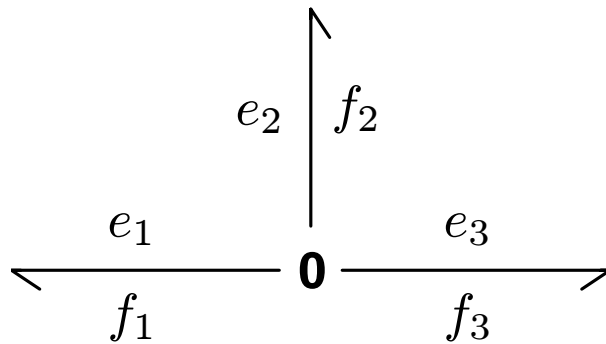
$$e_1 = e_2 = e_3$$

$$f_1 + f_2 + f_3 = 0$$

It is power continuous:

$$-e_1 f_1 - e_2 f_2 - e_3 f_3 = 0$$

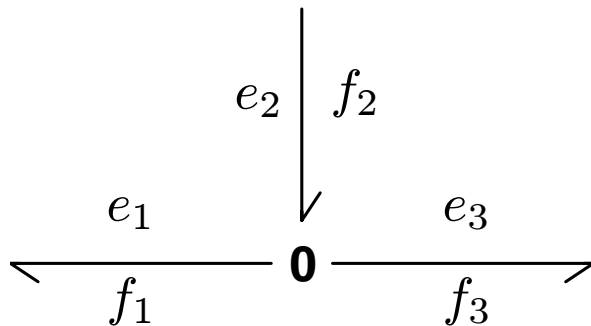Signs depend on power convention!

For instance, if

would still be
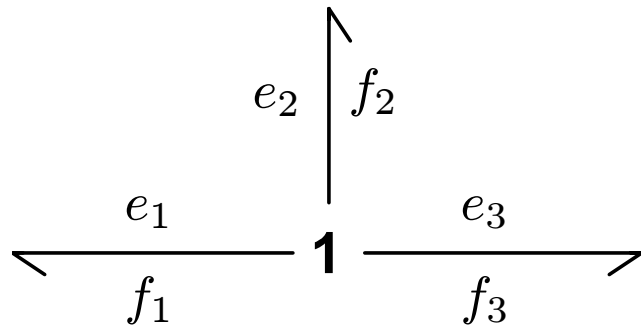
$$e_1 = e_2 = e_3$$

but

$$f_1 - f_2 + f_3 = 0$$

and

$$-e_1 f_1 + e_2 f_2 - e_3 f_3 = 0$$

1-junctions

1-junction relations are dual to those of 0-junctions:

$$f_1 = f_2 = f_3$$

$$e_1 + e_2 + e_3 = 0$$

Again, this is power continuous:

$$-e_1 f_1 - e_2 f_2 - e_3 f_3 = 0$$

0- and 1-junctions with an arbitrary number of bonds can be considered.
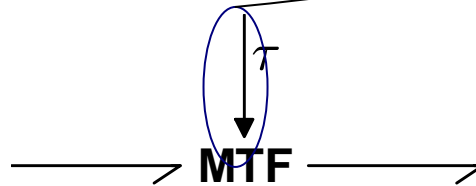
Notice that something like

    can be simplified to

but

    cannot be simplified

Some elements can be modulated.

This means that their parameters or constitutive relations
may depend on an external signal, carrying no power.

In bond graph theory, this is represented by an activated bond.

For instance, a modulated transformer is represented by

$$\tau$$

MTF

Activated bonds appear frequently in 2D and 3D mechanical
systems, and when representing instruments.

Special values of the modulus are represented with special symbols.
For instance, a gyrator with $\tau = 1$ is represented by

SGY

Flow sources, transformers and I-type elements
can be replaced by combinations of the other elements,
given rise to generalized bond graphs.

For instance,

$$\longrightarrow \mathbf{I} \qquad \text{is equivalent to} \qquad \longrightarrow \mathbf{GY} \longrightarrow \mathbf{C}$$
$$\quad\quad \Phi_I \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \tau \qquad\qquad \Phi_C$$

with

$$\tau q = p \qquad\qquad \Phi_C^{-1}(q) = \tau \Phi_I^{-1}(\tau q)$$

Nevertheless, we will use them to keep things simpler.

Generalized bond graphs are, however, necessary
in order to make contact with port-Hamiltonian theory.

Energy relations

For any element with a bond with power variables e and f, the energy variation from t_0 to t is

$$H(t) - H(t_0) = \int_{t_0}^{t} e(\tau) f(\tau) \, \mathrm{d}\tau$$

For C-type elements, e is a function of q and $\dot{q} = f$.

Changing variables from t to q,

$$H(q) - H(q_0) = \int_{q_0}^{q} \Phi_C^{-1}(\tilde{q}) \, \mathrm{d}\tilde{q}$$

In the linear case,

$$H(q) - H(q_0) = \frac{1}{2C} q^2 - \frac{1}{2C} q_0^2$$

For I-type elements, f is a function of p and $\dot{p} = e$.

Changing variables from t to p,
$$H(p) - H(p_0) = \int_{p_0}^{p} \Phi_I^{-1}(\tilde{p})\ \mathrm{d}\tilde{p}$$

In the linear case,
$$H(p) - H(p_0) = \frac{1}{2I}p^2 - \frac{1}{2I}p_0^2$$

For R-type elements, $e = \Phi_R(f)$ or $f = \Phi_R^{-1}(e)$. Then

$$H(t) - H(t_0) = \int_{t_0}^{t} \Phi_R(f(\tau))f(\tau)\ \mathrm{d}\tau = \int_{t_0}^{t} e(\tau)\Phi_R^{-1}(e(\tau))\ \mathrm{d}\tau$$

If the R-element is a true dissipator, $H(t) - H(t_0) \leq 0$, $\forall\ t \geq t_0$.

This means that the graph of Φ_R must be
completely contained in the first and third quadrant.

Causality

A bond links two elements, one of which
sets the effort and the other one the flow.

The causality assigment procedure chooses who sets what for each bond.

Causality assigment is necessary to transform
the bond graph into computable code.

For each bond, causality is indicated by the causal stroke.

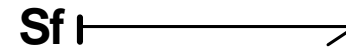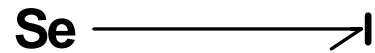A $\longrightarrow$ B   means that A sets e and B sets f

A $\longrightarrow$ B   means that B sets e and A sets f

Elements with fixed causality

Sources set either the effort or the flow, so only a causality is possible:

In gyrators and transformers, the variable
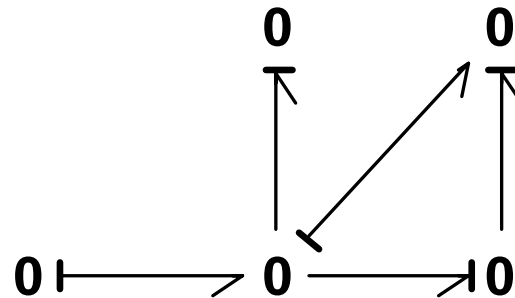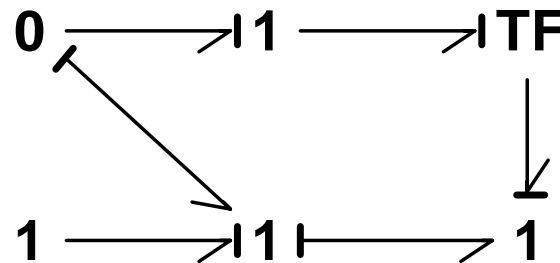relations allow only two causalities:

For 0-junctions, one of the bonds sets the effort
for the rest, so only one causal stroke is on the junction, while
the others are away from it:

For 1-junctions, one of the bonds sets the flow
for the rest, and its effort is computed from them, so all but one
of the causal strokes are on the junction, while
the remaining one is away from it:

Elements with preferred causality

Energy-storing elements, I or C, have a preferred causality, associated to the computation involving integrals instead of derivatives.

$$\vdash\!\!\longrightarrow \quad \textbf{C} \qquad\qquad \longrightarrow\!\!\shortmid\shortmid \quad \textbf{I}$$

This is called integral causality.

C-elements are given the flow and return the effort.

I-elements are given the effort and return the flow.

Differential causality is possible but not desirable:

{ Differentiation with respect to time implies knowledge of the future.
With differential causality, the response to an step input is unbounded.

Sometimes it is unavoidable and implies a reduction of state variables.

Elements with indifferent causality

R-type elements have, in principle, a causality
which can be set by the rest of the system:

$$f = \Phi_R^{-1}(e)$$

$$e = \Phi_R(f)$$

However, difficulty in writting either Φ_R or Φ_R^{-1}
may favor one of the two causalities.

For instance, in mechanical ideal Coulomb friction, F can be
expressed as a function of v, but not the other way around.

Mechanical domain example

General rules:

Each velocity is associated with a 1-junction,
including a reference (inertial) one.

Masses are linked as I-elements to the corresponding 1-junctions.
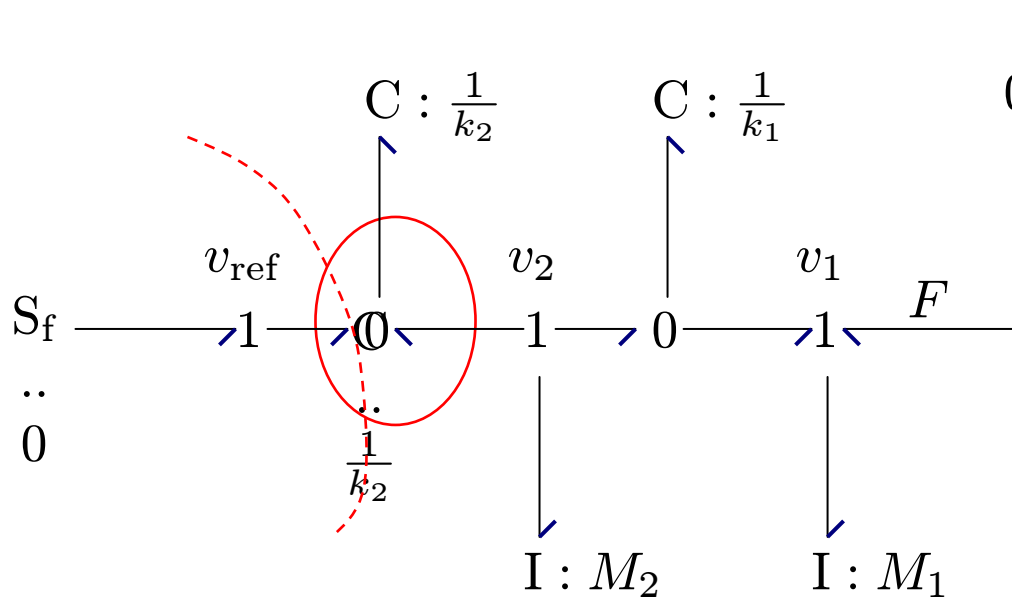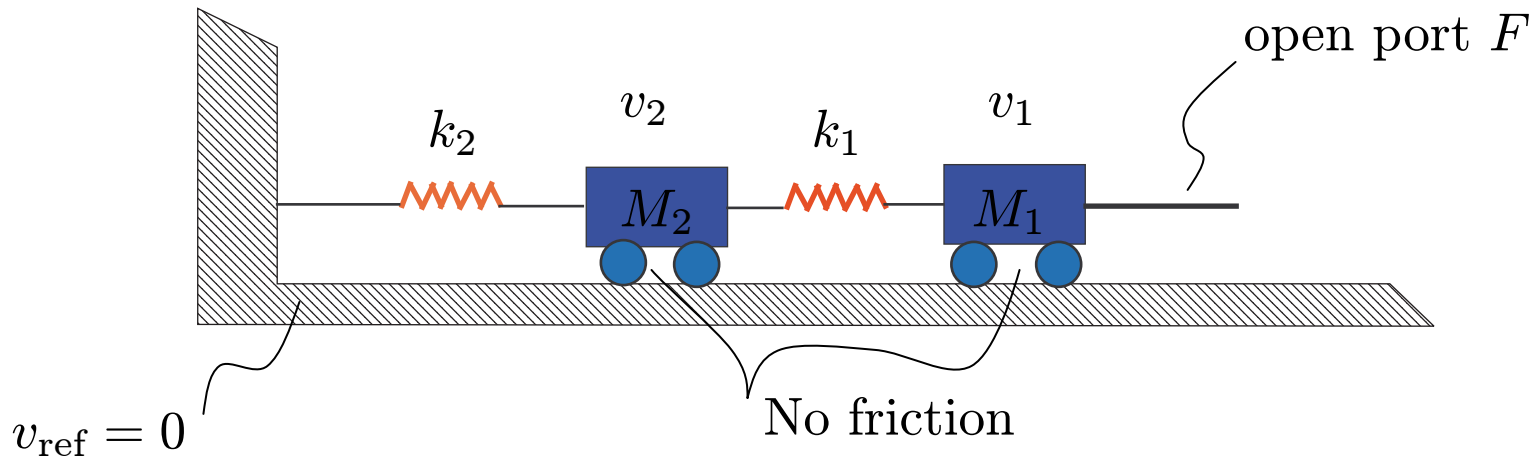
Springs and dissipative elements are linked to 0-junctions
connecting appropriate 1-junctions.

The rest of elements are inserted and power orientations are choosen.

The reference velocity is eliminated.

The bond graph is simplified.

Causality is propagated.

open port F

k_2 v_2 k_1 v_1

M_2 M_1

$v_{\mathrm{ref}} = 0$

No friction

power orientation
0-velocity reference
simplification

$C : \frac{1}{k_2}$ $C : \frac{1}{k_1}$

v_{ref} v_2 v_1

$\mathrm{S_f}$ ——— 1 —— 0 ←—— 1 —— 0 —— 1 ←—— F

$\ddot{0}$

$\frac{1}{k_2}$

$\mathrm{I} : M_2$ $\mathrm{I} : M_1$

The final (acausal) bond graph is thus
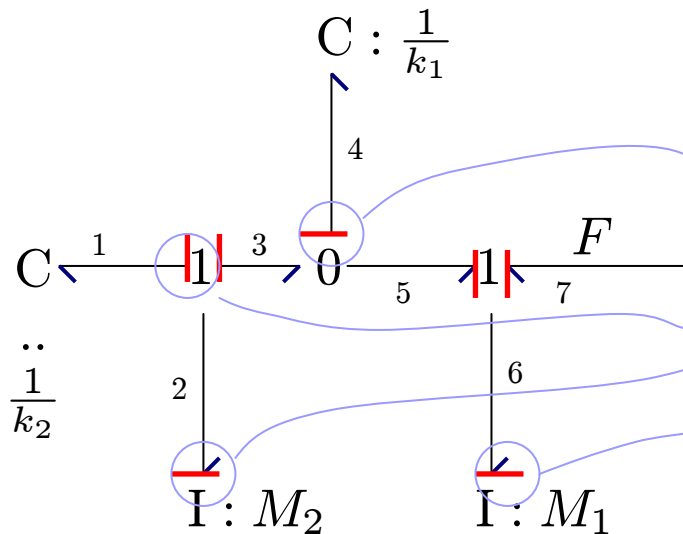
Causality propagation

Hence, all the storage elements get an integral causality assignation.
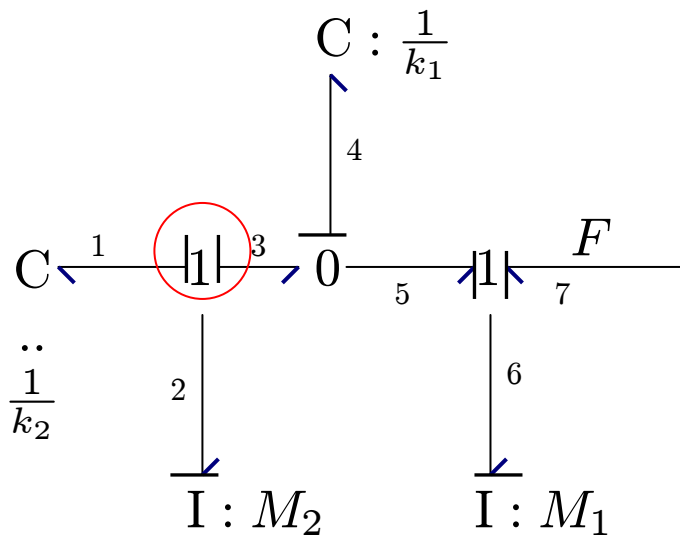
Finally, we assign numbers to the bonds.

For each storage element, the state variable will be designed with the same index as the bond.

$$f_1 = f_2 = f_3 \qquad e_2 = -e_1 - e_3$$

$$e_3 = e_4 = e_5 \qquad f_4 = f_3 - f_5$$

$$f_5 = f_6 = f_7 \qquad e_6 = e_5 + e_7$$

$$\dot{q}_1 = f_1 \qquad e_1 = k_2 q_1$$

$$\dot{p}_2 = e_2 \qquad f_2 = \frac{1}{M_2} p_2$$

$$\dot{q}_4 = f_4 \qquad e_4 = k_1 q_4$$

$$\dot{p}_6 = e_6 \qquad f_6 = \frac{1}{M_1} p_6$$

$$e_7 = F$$

$$\dot{q}_1 = f_1 = f_2 = \frac{1}{M_2} p_2 \quad (= v_2)$$

$$\dot{p}_2 = e_2 = -e_1 - e_3 = -k_2 q_1 - e_4 = -k_2 q_1 - k_1 q_4$$

$$\dot{q}_4 = f_4 = f_3 - f_5 = f_2 - f_6 = \frac{1}{M_2} p_2 - \frac{1}{M_1} p_6 \quad (= v_2 - v_1)$$

$$\dot{p}_6 = e_6 = e_5 + e_7 = e_4 + F = k_1 q_4 + F$$

System of ODE
for analysis
and simulation

Energy balance

$$H(q_1, p_2, q_4, p_6) = \tfrac{1}{2}k_2 q_1^2 + \tfrac{1}{2}k_1 q_4^2 + \frac{1}{2M_2}p_2^2 + \frac{1}{2M_1}p_6^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H = k_2 q_1 \dot{q}_1 + k_1 q_4 \dot{q}_4 + \frac{1}{M_2}p_2 \dot{p}_2 + \frac{1}{M_1}p_6 \dot{p}_6$$

$$= k_2 q_1 \left(\frac{1}{\cancel{M_2}}p_2 \right) + k_1 q_4 \left(\frac{1}{\cancel{M_2}}p_2 - \cancel{\frac{1}{M_1}p_6} \right) \qquad \dot{H} = \frac{1}{M_1}p_6 F = v_1 F$$

$$+ \frac{1}{M_2}p_2 \left(-\cancel{k_2 q_1} - \cancel{k_1 q_4} \right) + \frac{1}{M_1}p_6 \left(\cancel{k_1 q_4} + F \right)$$

Since the spring k_2 is to the left of the mass M_2, it follows from $\dot{q}_1 = v_2$ that v_2 is positive to the right.

Similarly, since the spring k_1 is to the left of M_1, it follows from $\dot{q}_4 = v_2 - v_1$ that v_1 is positive to the left.

Finally, from the later and $\dot{p}_6 = k_1 q_1 + F$ one deduces that F is positive to the left.

Hence, v_1 and F have the same positive orientation and $v_1 F$ is the power into the system.
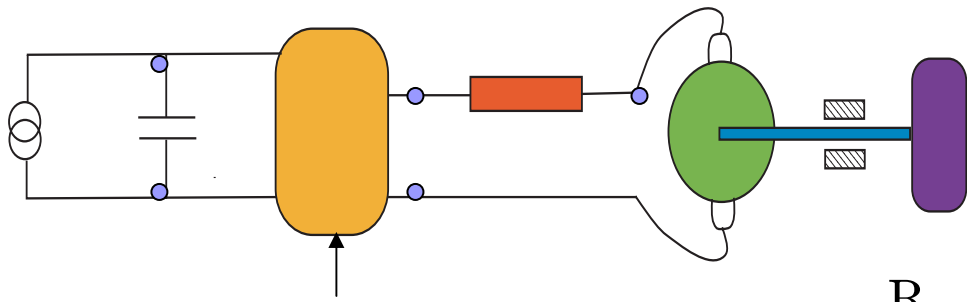
Multidomain example
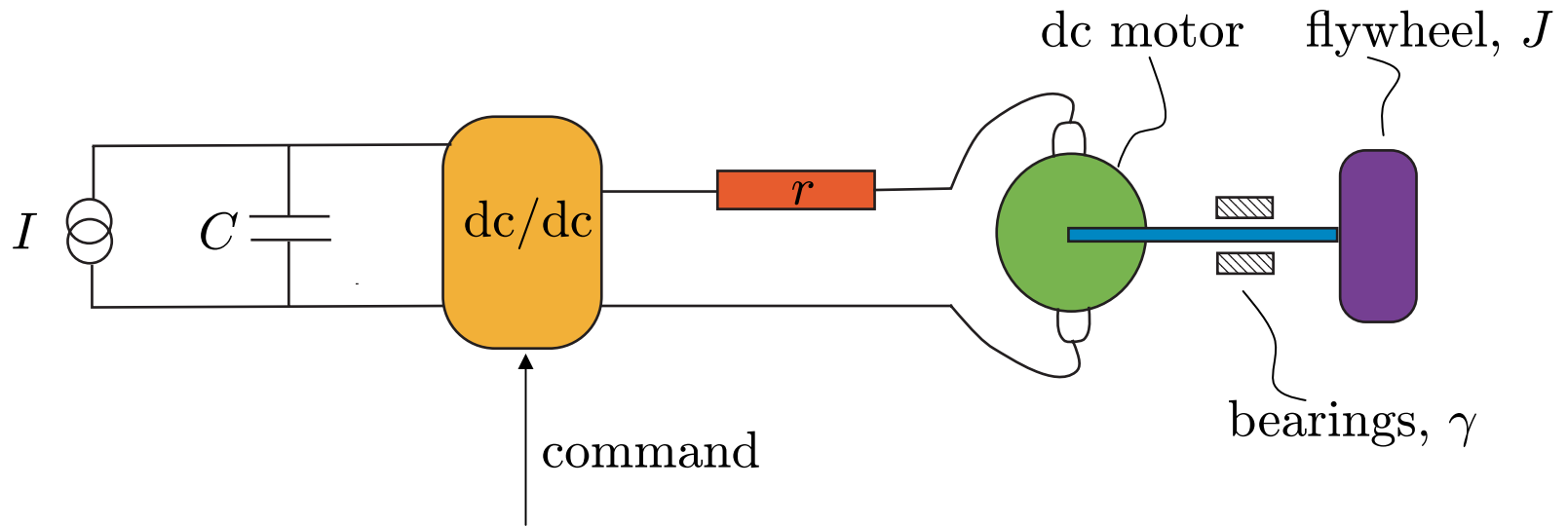
We will model the converter as a modulated transformer, and the dc motor as a gyrator.
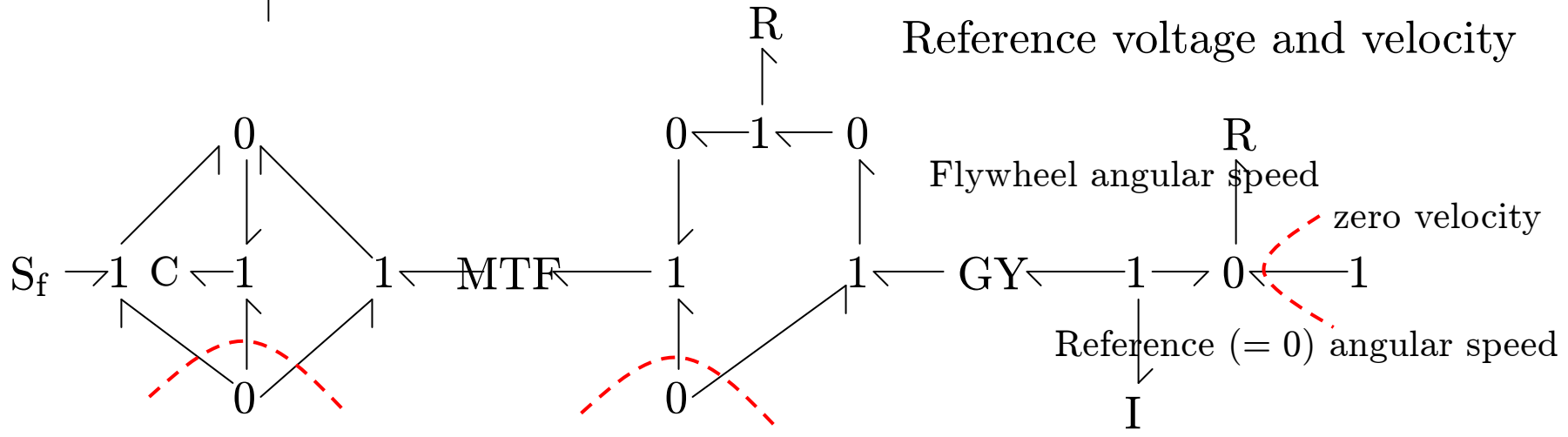
In the electrical domain, a 0-junction is introduced for each voltage, and everything is connected in between by means of 1-junctions.

In the electrical domain
$$\begin{cases} \text{0-junction} \equiv \text{parallel connection} \\ \text{1-junction} \equiv \text{series connection} \end{cases}$$

Voltage nodes
Electric elements insertion
Velocities
Flywheel   Friction
Power convention
Reference voltage and velocity

R

$0 \leftarrow 1 \leftarrow 0$

R

Flywheel angular speed

zero velocity

$S_f \rightarrow 1 \; C \leftarrow 1 \qquad 1 \leftarrow MTF \qquad 1 \qquad 1 \leftarrow GY \leftarrow 1 \rightarrow 0 \leftarrow 1$

0 0

Reference (= 0) angular speed

I

We set to earth these two

After eliminating these three nodes and their bonds, several
simplifications can be carried out.

The final bond graph, with causal assignment and bond naming, is

$$R : r$$

$$S_f \xrightarrow{\;1\;} 0 \xrightarrow{\;3\;} MTF \xleftarrow{\;4\;} 1 \xleftarrow{\;6\;} GY \xleftarrow{\;7\;} 1 \xrightarrow{\;9\;} I : J$$

$$\ddot{I}$$

$$\downarrow 2$$

$$\tau(t)$$

$$C : C$$

$$5$$

$$g$$

$$\downarrow 8$$

$$R : \gamma$$

$$e_4 = e_3 \frac{1}{\tau(t)} \qquad\qquad e_6 = g f_7$$

$$f_3 = f_4 \frac{1}{\tau(t)} \qquad\qquad e_7 = g f_6$$
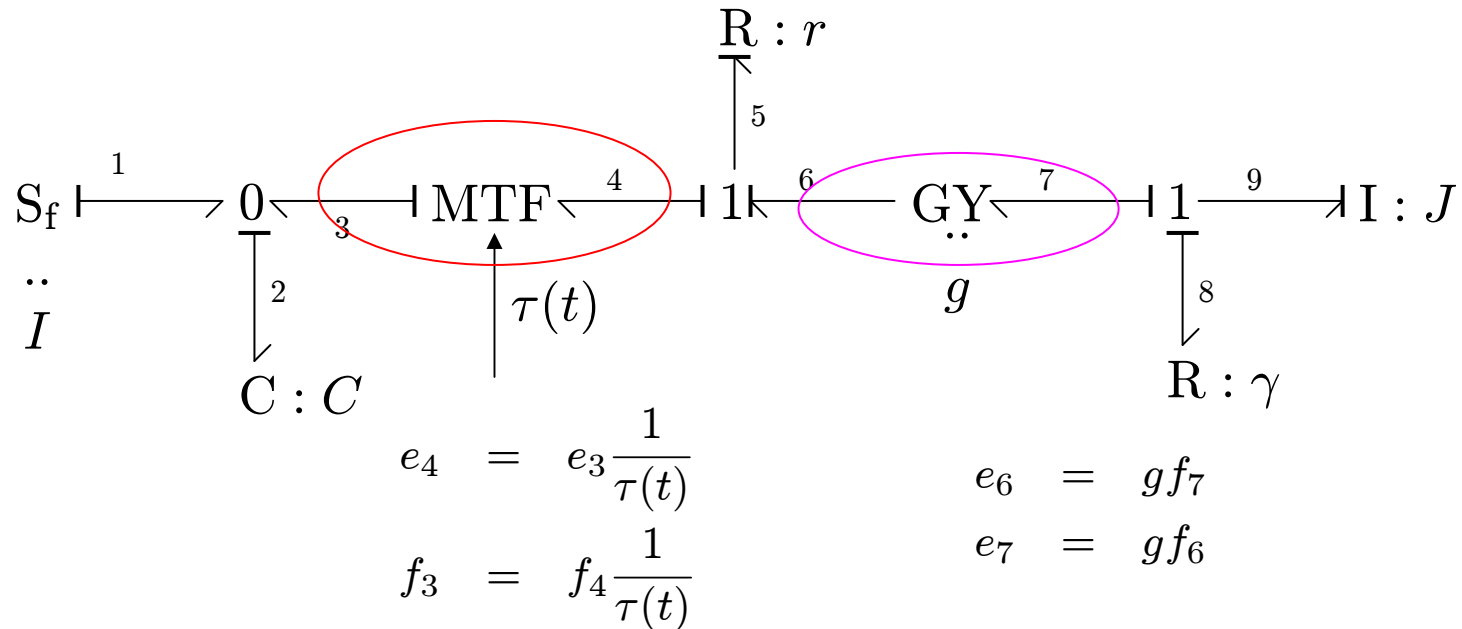
Exercise

Write all the network and constitutive relations

Obtain the state space equations

Write down the energy balance equation

Next seminar

- Storage and dissipation elements with several ports.
- Thermodynamic systems.
- Dirac structures and bond graphs.
- Distributed systems.